A Bose-Laskar-Hoffman theory for μ-bounded graphs with fixed smallest eigenvalue

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Abstract

In 2018, by Ramsey and Hoffman theory, Koolen, Yang, and Yang presented a structural result on graphs with smallest eigenvalue at least -3 and large minimum degree. In this study, we depart from the conventional use of Ramsey theory and instead employ a novel approach that combines the Bose-Laskar type argument with Hoffman theory to derive structural insights into μ-bounded graphs with fixed smallest eigenvalue. Our method establishes a reasonable bound on the minimum degree. Note that local graphs of distance-regular graphs are μ-bounded. We apply these results to characterize the structure for any local graph of a distance-regular graph with classical parameters (D, b, α, β). Consequently, we show that the parameter α is bounded by a cubic polynomial in b if $D \geq 9$ and $b \geq 2$. We also show that $\alpha \leq 2$ if $b = 2$ and $D \geq 12$.

Keywords: Bose-Laskar method; distance-regular graphs; Hoffman theory; μ-bounded graphs
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1 Introduction

All graphs mentioned in this paper are finite, undirected and simple. The eigenvalues of a graph are the eigenvalues of its adjacency matrix. Let G be a graph. We denote by $\lambda_{\min}(G)$ the smallest eigenvalue of G, $N(x)$ the neighbors of vertex x, and $d(x)$ the degree of vertex x. For a non-negative integer a, let $[a] := \{1, \ldots, a\}$ and $[0] := \emptyset$. For undefined notations, see [2].

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A graph is called *μ-bounded* with parameter c if each pair of distinct non-adjacent vertices has at most c common neighbors. Note that the local graph of distance-regular graphs at every vertex is μ-bounded. The notions of distance-regular graphs and distance-regular graphs with classical parameters are introduced in Section 2.

In this paper, we will give some structural results on μ-bounded graphs with fixed smallest eigenvalue. After that, we apply these results to describe the structure for any local graph of a distance-regular graph with classical parameters (D, b, α, β). As a consequence, Theorem 1.1 gives an upper bound on α in terms of b. Although the bound is similar to the bound in [12], the main difference here is that we derive a reasonable bound on the diameter, whereas in [12], they need a super large bound on the diameter.

Theorem 1.1. *Let D and b be positive integers. If Γ is a distance-regular graph with classical parameters (D, b, α, β) such that $D \geq 9$ and $b \geq 2$, then $\alpha < b^2(b+1)+1$ holds.*

In particular, Theorem 1.2 shows that $\alpha \leq 2$ if $b = 2$ and $D \geq 12$.

Theorem 1.2. *If Γ is a distance-regular graph with classical parameters (D, b, α, β) such that $b = 2$ and $D \geq 12$, then $\alpha \leq 2$ and $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, 2\}$.*

Moreover, we can show the following bound for α if $b \in [100]$.

Theorem 1.3. *If Γ is a distance-regular graph with classical parameters (D, b, α, β) such that $b \in [100]$ and $D \geq 14$, then the following statements hold:*

(i) *If b is not a perfect square, then $\alpha \leq b$.*

(ii) *If b is a perfect square, then $\alpha \leq b$ or $\alpha = b + \sqrt{b}$.*

It is worth mentioning that the half dual polar graph and the distance 1-or-2 graph of symplectic dual polar graph are distance-regular graphs with classical parameters (D, b, α, β) such that $b = q^2$ and $\alpha = q(q+1) = b + \sqrt{b}$, where q is a prime power. For definition of these two kinds of distance-regular graphs, we refer readers to [2, Chapter 9.4].

Conjecture 1.4. *Let Γ be a distance-regular graph with classical parameters (D, b, α, β). There is a constant ℓ such that if $D \geq \ell$, then one of the following statements holds:*

(i) *If b is a perfect square, then $\alpha \leq b + \sqrt{b}$.*

(ii) *If b is not a perfect square, then $\alpha \leq b$.*

In 1976, Cameron et al. [3] showed that a connected graph G with smallest eigenvalue $\lambda_{\min}(G) \geq -2$ is a generalized line graph if the number of vertices is at least 37, by using the classification of the irreducible root lattices. In 1977, Hoffman [6] showed that a connected graph G with smallest eigenvalue $\lambda_{\min}(G) > -1-\sqrt{2}$ is a generalized line graph if the minimum degree of G is large enough. Hoffman did not use the classification of the irreducible root lattices for the proof, but instead he had to assume large minimum degree. In terms of line Hoffman graphs, a generalized line graph is the slim graph of a {  } -line Hoffman graph. The notions of Hoffman graphs and line Hoffman graphs are introduced in Section 2, and the notion of associated Hoffman graphs in Section 3.

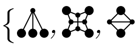
In 2018, Koolen et al. [15] showed the following result on graphs with smallest eigenvalue at least -3, which generalized the result by Hoffman [6]. To state the result, we need to introduce the concept of a t-plex. Given a positive integer t, a *t-plex* is a graph such that each vertex is adjacent to all but at most $p-1$ of the other vertices. Especially, a *clique* is a 1-plex. The notions of Hoffman graphs, line Hoffman graphs, and slim graphs mentioned below will be introduced in Sections 2.2 and 2.3.

Theorem 1.5 ([15, Theorem 1.4])**.** *There exists a positive integer K such that if a graph G satisfies the following conditions:*

(i) *$d(x) > K$ for each $x \in V(G)$,*

(ii) *any 5-plex containing x has order at most $d(x) - K$ for all $x \in V(G)$,*

(iii) *$\lambda_{\min}(G) \geq -3$,*

then G is the slim graph of a 2-fat {  *}-line Hoffman graph.*

For the proof, it is important to have large cliques. To obtain those large cliques in the proof of Theorem 1.5, they needed to use Ramsey theory, implying that they do not have a reasonable bound for K.

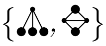
By combining the Bose-Laskar type argument and Hoffman theory, we give the following result which is a version of Theorem 1.5 for μ-bounded graphs. As a consequence, we derive a reasonable value for K in this case. The main reason that we can obtain a reasonable bound for K is Lemma 4.1. This lemma guarantees the existence of large cliques without relying on Ramsey theory which plays a key role in the proof of Theorem 1.6.

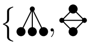
Theorem 1.6. *Let c be a positive integer. Let $\tilde{c} := \min\{c, 6\}$ and $q := \max\{c+5, 50\tilde{c}+16\}$. If a graph G satisfies the following conditions:*

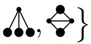
(i) *G is a μ-bounded graph with parameter c,*

(ii) *every clique C has order at most $\min_{x\in V(C)} d(x) - K$, where $K = \max\{36c+400\tilde{c}+83, 44c-5\}$,*

(iii) $\lambda_{\min}(G) \geq -3$,

then the associated Hoffman graph $\mathfrak{g}(G,q)$ *is a* 2*-fat* $\{$$\}$ *-line Hoffman graph.*

Notice that if a μ-bounded graph G with parameter c is the slim graph of a 2-fat $\{$$\}$-line Hoffman graph, then its structure is characterized in Propositions 3.2 and 3.3, and thus $c \leq 18$ and $\lambda_{\min}(G) \geq -3$. In particular, $c \leq 12$ if G is regular.

In fact, for fixed $\epsilon > 0$, the condition (iii) of Theorem 1.6 can be generalized to $\lambda_{\min}(G) \geq -2-\sqrt{2}+\epsilon$ (see Theorem 4.3), and the value of K in (ii) would depend on ϵ. It is worth mentioning that the condition $\epsilon > 0$ is necessary. For example, let $H := L(K_{p,p})\square K_{1,2}$ be the Cartesian product of $L(K_{p,p})$ and $K_{1,2}$, where $L(K_{p,p})$ is the line graph of complete bipartite graph $K_{p,p}$. It follows that H is a μ-bounded graph with parameter 2, every vertex lies in two distinct cliques of order p, and $\lambda_{\min}(H) = -2-\sqrt{2}$ and p can be arbitrary large. But H is not the slim graph of a 2-fat $\{$$\}$-line Hoffman graph.

As an application of Theorem 1.6, we give a characterization for the structure of local graphs for some distance-regular graphs with classical parameters. In particular, Theorem 1.2 is derived from Corollary 1.7.

Corollary 1.7. *Let* Γ *be a distance-regular graph with classical parameters* (D,b,α,β) *such that* $b=2$ *and* $\alpha \neq 0$*. Let* $c := 3\alpha+2$*,* $\tilde{c} := \min\{c,6\}$ *and* $q := \max\{c+5, 50\tilde{c}+16\}$*. For each vertex* x*, the local graph* Γ_x *at* x *is* μ*-bounded with parameter* c *and the associated Hoffman graph* $\mathfrak{g}(\Gamma_x,q)$ *is a* 2*-fat* $\{$$\}$*-line Hoffman graph if* $\alpha(2^D-2) \geq \max\{36c+400\tilde{c}+84, 44c-4\}$*.*

Remark 1.8. *If* Γ *is a distance-regular graph with classical parameters* $(D,2,2,2^{D+2}-2)$*, then the above result provides structural insight for any local graph of* Γ*. On this moment there are two infinite families known with these parameters, namely the Grassmann graphs* $J_2(2D+1,D)$ *and the twisted Grassmann graphs* $\tilde{J}_2(2D+1,D)$*, but we do not know whether these are the only distance-regular graphs with these parameters for large* D*.*

This paper is organized as follows. In Section 2, we give definitions and basic theory of Hoffman graphs. We also give some results about the forbidden matrices introduced by Koolen et al. [15]. In Section 3, we give a theory for the associated Hoffman graph of μ-bounded graphs. This is a simplified version of the original method of Hoffman as given in [9]. We also show some properties for associated Hoffman graphs. In Section 4, we firstly give the crucial Lemma 4.1 with an argument inspired by the Bose-Laskar method. After that, we give a proof for the main result Theorem 1.6. In Section 5, we give some applications on distance-regular graphs with classical parameters and prove Corollary 1.7, Theorem 1.2, and Theorem 1.3. We also give a proof of Theorem 1.1. This will be a consequence of Theorem 5.2, whose proof will be also given in that section.

2 Definitions and preliminaries

2.1 Distance-regular graphs

A connected graph Γ with diameter D is called *distance-regular* if for $0 \le i \le D$ there are integers b_i and c_i such that for every pair of vertices $x, y \in V(\Gamma)$ with $d(x, y) = i$, among the neighbors of y, there are precisely c_i (resp. b_i) at distance $i-1$ (resp. $i+1$) from x. This pivotal concept led to significant results in graph theory and algebraic combinatorics [2].

In this paper, we denote a distance-regular graph by Γ. Every distance-regular graph is regular with degree $k = b_0$. We define $a_i := k - b_i - c_i$ for notational convenience. Note that local graphs of Γ are a_1-regular and μ-bounded with parameter $c_2 - 1$ on b_0 vertices. The numbers c_i, a_i, and b_i $(0 \le i \le D)$ are called the *intersection numbers* of Γ with the following property.

Proposition 2.1 ([2, Theorem 4.1.7])**.** *If i and h are positive integers such that $i + h \le D$, then $p^{i+h}_{ih} := \frac{c_{i+1}\cdots c_{i+h}}{c_1 \cdots c_h}$ is a non-negative integer.*

Here we introduce the Delsarte bound for distance-regular graphs, which gives an upper bound on the order of cliques.

Lemma 2.2 ([2, Proposition 4.4.6])**.** *Let Γ be a distance-regular graph with degree k and smallest eigenvalue $\lambda_{\min}(\Gamma)$. If C is a clique in Γ, then the order of the clique satisfies*

$$|V(C)| \le 1 + \frac{k}{-\lambda_{\min}(\Gamma)}.$$

The following theorem is a well-known result for the smallest eigenvalue of local graphs of distance-regular graphs, and is due to Terwilliger [20].

Theorem 2.3 ([2, Theorem 4.4.3])**.** *If Γ is a distance-regular graph of diameter $D \ge 3$ with second largest eigenvalue θ, then $b^+ := \frac{b_1}{\theta+1} > 0$, and the local graph of Γ at each vertex has smallest eigenvalue at least $-1 - b^+$.*

Definition 2.4. *A distance-regular graph Γ of diameter D has* classical parameters (D, b, α, β) *if the intersection numbers of Γ satisfy*

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_b \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_b\right) \tag{1}$$

and

$$b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix}_b - \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right) \tag{2}$$

for $0 \le i \le D$, where $\begin{bmatrix} i \\ 1 \end{bmatrix}_b = b^{i-1} + b^{i-2} + \cdots + b + 1$.

There are many examples of distance-regular graphs with classical parameters such as the Johnson graphs, the Hamming graphs, the Grassmann graphs, the bilinear forms graphs and so on. For more comprehensive background on distance-regular graphs, we refer readers to [2, 21].

Brouwer et al. [2, Proposition 6.2.1] showed that the parameter b is an integer such that $b \notin \{0, -1\}$. Weng [22] classified the distance-regular graphs with classical parameters (D, b, α, β) for $b \leq -2$. Terwilliger, referred to [2, Theorem 6.1.1], has classified the distance-regular graphs with classical parameters (D, b, α, β), where $b = 1$. Therefore, we focus on $b \geq 2$ in this paper.

The following lemma gives explicit formulas for eigenvalues of distance-regular graphs with classical parameters.

Lemma 2.5 ([2, Corollary 8.4.2])**.** *The eigenvalues of the distance-regular graph Γ with classical parameters (D, b, α, β) are $\lambda_i := \begin{bmatrix} D-i \\ 1 \end{bmatrix}_b (\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}_b) - \begin{bmatrix} i \\ 1 \end{bmatrix}_b = \frac{b_i}{b^i} - \begin{bmatrix} i \\ 1 \end{bmatrix}_b$ for $0 \leq i \leq D$.*

Note that, if $b \geq 1$, then these eigenvalues have natural ordering $k = \lambda_0 > \lambda_1 > \cdots > \lambda_D$. Lemma 2.2 and Lemma 2.5 imply that the Delsarte bound for distance-regular graphs with classical parameters is $1 + \beta$ if $b \geq 1$. We also have

$$\beta - 1 \geq \alpha \begin{bmatrix} D-1 \\ 1 \end{bmatrix}_b, \tag{3}$$

since $a_D = k - b_D - c_D = \begin{bmatrix} D \\ 1 \end{bmatrix}_b \beta - 0 - \begin{bmatrix} D \\ 1 \end{bmatrix}_b (1 + \alpha \begin{bmatrix} D-1 \\ 1 \end{bmatrix}_b) \geq 0$.

2.2 Hoffman graphs and special matrices

Now we give definitions and preliminaries of Hoffman graphs. For more details, see [8, 9, 15, 23].

Definition 2.6. *A* Hoffman graph $\mathfrak{h}$ *is a pair of (H, ℓ), where $H = (V, E)$ is a graph and $\ell : V \to \{\text{fat}, \text{slim}\}$ is a labeling map on V, such that no two vertices with label* fat *are adjacent and every vertex with label* fat *has at least one neighbor with label* slim*.*

The vertices with label fat (resp. slim) are called *fat vertices* (resp. *slim vertices*). We denote the set of slim (resp. fat) vertices by $V_{\text{slim}}(\mathfrak{h})$ (resp. $V_{\text{fat}}(\mathfrak{h})$). For a vertex x of $\mathfrak{h}$, the set of slim (resp. fat) neighbors of x in $\mathfrak{h}$ is denoted by $N_{\mathfrak{h}}^{\text{slim}}(x)$ (resp. $N_{\mathfrak{h}}^{\text{fat}}(x)$). A Hoffman graph is called *t-fat* if every slim vertex has at least t fat neighbors. The *slim graph* of $\mathfrak{h}$ is the subgraph of H induced by $V_{\text{slim}}(\mathfrak{h})$. Note that the slim graph of a Hoffman graph is an ordinary graph.

Definition 2.7. *A Hoffman graph $\mathfrak{h}_1 = (H_1, \ell_1)$ is called an* induced Hoffman subgraph *of $\mathfrak{h} = (H, \ell)$, if H_1 is an induced subgraph of H and $\ell_1(x) = \ell(x)$ for all vertices x of H_1.*

Definition 2.8. *Two Hoffman graphs $\mathfrak{h} = (H, \ell)$ and $\mathfrak{h}' = (H', \ell')$ are* isomorphic *if there exists an isomorphism from H to H' which preserves the labeling.*

For a Hoffman graph $\mathfrak{h} = (H, \ell)$, there exists a matrix D such that the adjacency matrix A of H satisfies $A = \begin{pmatrix} A_{\text{slim}} & D^T \\ D & O \end{pmatrix}$, where A_{slim} is the adjacency matrix of the slim graph of $\mathfrak{h}$. The symmetric matrix $S(\mathfrak{h}) := A_{\text{slim}} - D^T D$ is called the *special matrix* of $\mathfrak{h}$. The eigenvalues of $\mathfrak{h}$ are the eigenvalues of $S(\mathfrak{h})$, and the smallest eigenvalue of $\mathfrak{h}$ is denoted by $\lambda_{\min}(\mathfrak{h})$. Woo and Neumaier[23, Corollary 3.3] showed that $\lambda_{\min}(\mathfrak{g}) \geq \lambda_{\min}(\mathfrak{h})$ if $\mathfrak{g}$ is an induced Hoffman subgraph of $\mathfrak{h}$.

The t-fat Hoffman graph with one slim vertex and t fat vertices, denoted by $\mathfrak{h}^{(t)}$, is the unique Hoffman graph with the special matrix $(-t)$. However, in general Hoffman graphs are not determined by their special matrices. For example,  and  are non-isomorphic Hoffman graphs with the same special matrix.

Let $\mathfrak{h}$ be a Hoffman graph. Let $G(\mathfrak{h}, p)$ denote the graph obtained from $\mathfrak{h}$ by replacing fat vertex f by a slim p-clique K^f and joining all the neighbors of f in $\mathfrak{h}$ with all the vertices in K^f for each fat vertex f. For the relation between $\lambda_{\min}(\mathfrak{h})$ and $\lambda_{\min}(G(\mathfrak{h}, p))$, Hoffman and Ostrowski showed the following result. For a proof, we refer to [8, Theorem 2.14] and [14, Theorem 3.4].

Theorem 2.9. *If $\mathfrak{h}$ is a Hoffman graph and p a positive integer, then $\lambda_{\min}(G(\mathfrak{h}, p)) \geq \lambda_{\min}(\mathfrak{h})$ and $\lim_{p\to\infty} \lambda_{\min}(G(\mathfrak{h}, p)) = \lambda_{\min}(\mathfrak{h})$.*

2.3 Decompositions of Hoffman graphs

Now, we introduce decompositions of Hoffman graphs and line Hoffman graphs.

For two vertices x and y, we denote by $x \sim y$ if they are adjacent. Let W be a subset of $V_{\text{slim}}(\mathfrak{h})$. The *induced Hoffman subgraph of $\mathfrak{h}$ generated by W* is the Hoffman subgraph of $\mathfrak{h}$ induced by $W \cup \{f \in V_{\text{fat}}(\mathfrak{h}) \mid f \sim w \text{ for some } w \in W\}$.

Definition 2.10. *A family of Hoffman graphs $\{\mathfrak{h}^i\}_{i=1}^r$ is a* decomposition *of a Hoffman graph $\mathfrak{h}$, if there exists a partition $\left\{V_{\text{slim}}^1(\mathfrak{h}), V_{\text{slim}}^2(\mathfrak{h}), \ldots, V_{\text{slim}}^r(\mathfrak{h})\right\}$ of $V_{\text{slim}}(\mathfrak{h})$ such that the induced Hoffman subgraph generated by $V_{\text{slim}}^i(\mathfrak{h})$ is $\mathfrak{h}^i$ for $i \in [r]$ and*

$$S(\mathfrak{h}) = \begin{pmatrix} S(\mathfrak{h}^1) & & & \\ & S(\mathfrak{h}^2) & & \\ & & \ddots & \\ & & & S(\mathfrak{h}^r) \end{pmatrix}$$

is a block diagonal matrix with respect to this partition of $V_{\text{slim}}(\mathfrak{h})$.

If a Hoffman graph $\mathfrak{h}$ has a decomposition $\{\mathfrak{h}^i\}_{i=1}^r$, then we write $\mathfrak{h} = \uplus_{i=1}^r \mathfrak{h}^i$. A Hoffman graph $\mathfrak{h}$ is said to be *decomposable*, if $\mathfrak{h}$ has a decomposition $\{\mathfrak{h}^i\}_{i=1}^r$ with $r \geq 2$. Otherwise, $\mathfrak{h}$ is called *indecomposable*. Note that $\mathfrak{h}$ is decomposable if and only if its special matrix $S(\mathfrak{h})$ is a block diagonal matrix with at least two blocks. Here we give a combinatorial way to describe the decomposability of Hoffman graphs.

Lemma 2.11 ([15, Lemma 2.11])**.** *If $\mathfrak{h}$ is a Hoffman graph and $\{\mathfrak{h}^i\}_{i=1}^r$ a family of induced Hoffman subgraphs of $\mathfrak{h}$, then $\{\mathfrak{h}^i\}_{i=1}^r$ is a decomposition of $\mathfrak{h}$, if it satisfies the following conditions:*

(i) $V(\mathfrak{h}) = \bigcup_{i=1}^r V(\mathfrak{h}^i)$,

(ii) $V_{\text{slim}}(\mathfrak{h}^i) \cap V_{\text{slim}}(\mathfrak{h}^j) = \emptyset$ *for* $i \neq j$,

(iii) *if* $x \in V_{\text{slim}}(\mathfrak{h}^i)$, $f \in V_{\text{fat}}(\mathfrak{h})$ *and* $x \sim f$, *then* $f \in V_{\text{fat}}(\mathfrak{h}^i)$,

(iv) *if* $x \in V_{\text{slim}}(\mathfrak{h}^i)$, $y \in V_{\text{slim}}(\mathfrak{h}^j)$ *and* $i \neq j$, *then* x *and* y *have at most one common fat neighbor, and they have one if and only if they are adjacent.*

Definition 2.12. *Let $\mathfrak{G}$ be a family of pairwise non-isomorphic Hoffman graphs. A Hoffman graph $\mathfrak{h}$ is called a* $\mathfrak{G}$-line Hoffman graph *if there exists a Hoffman graph $\mathfrak{h}'$ satisfying the following conditions:*

(i) $\mathfrak{h}'$ *has* $\mathfrak{h}$ *as an induced Hoffman subgraph,*

(ii) $\mathfrak{h}'$ *has the same slim graph as* $\mathfrak{h}$,

(iii) $\mathfrak{h}' = \uplus_{i=1}^r \mathfrak{h}'_i$, *where* $\mathfrak{h}'_i$ *is isomorphic to an induced Hoffman subgraph of a Hoffman graph in* $\mathfrak{G}$ *for* $i \in [r]$.

2.4 Forbidden matrices

In this subsection, we introduce a structural theory for t-fat Hoffman graphs with smallest eigenvalue at least $-t-1$. For more details, we refer readers to [15]

Let t and a be integers, where $t > 0$. Let $m_{1,a}(t) := (-t+a)$, $m_{2,a}(t) := \begin{pmatrix} -t & a \\ a & -t \end{pmatrix}$, $m_{3,a}(t) := \begin{pmatrix} -t-1 & a \\ a & -t \end{pmatrix}$, $m_{4,a}(t) := \begin{pmatrix} -t-1 & a \\ a & -t-1 \end{pmatrix}$, $m_5(t) := \begin{pmatrix} -t & -1 & -1 \\ -1 & -t & -1 \\ -1 & -1 & -t \end{pmatrix}$, $m_6(t) := \begin{pmatrix} -t & 1 & 1 \\ 1 & -t & -1 \\ 1 & -1 & -t \end{pmatrix}$, $m_7(t) := \begin{pmatrix} -t & 0 & 1 \\ 0 & -t & -1 \\ 1 & -1 & -t \end{pmatrix}$, $m_8(t) := \begin{pmatrix} -t & 0 & 1 \\ 0 & -t & 1 \\ 1 & 1 & -t \end{pmatrix}$, and $m_9(t) := \begin{pmatrix} -t & 0 & -1 \\ 0 & -t & -1 \\ -1 & -1 & -t \end{pmatrix}$.

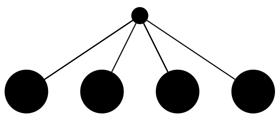
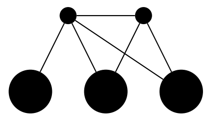
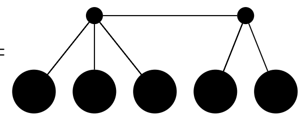
The set of irreducible symmetric matrices $\mathscr{M}(t)$ is the union of the sets $M_1(t), M_2(t)$, and $M_3(t)$, where $M_1(t) := \{m_{1,a}(t), m_{2,a}(t) \mid a = -2, -3, \ldots\}$, $M_2(t) := \{m_{3,a}(t), m_{4,a}(t) \mid a = 1, -1, -2, \ldots\}$, $M_3(t) := \{m_5(t), m_6(t), m_7(t), m_8(t), m_9(t)\}$.

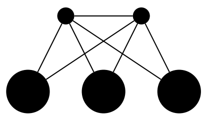
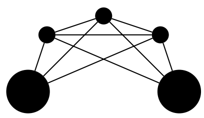
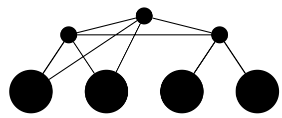
Note that $\lambda_{\min}(m_{1,a}(t)) = -t+a$, $\lambda_{\min}(m_{2,a}(t)) = -t-|a|$, $\lambda_{\min}(m_{3,a}(t)) = -t - \frac{1+\sqrt{1+4a^2}}{2}$, $\lambda_{\min}(m_{4,a}(t)) = -t-1-|a|$, $\lambda_{\min}(m_5(t)) = \lambda_{\min}(m_6(t)) = -t-2$, and $\lambda_{\min}(m_7(t)) = \lambda_{\min}(m_8(t)) =$

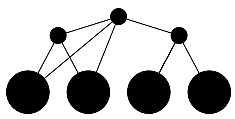
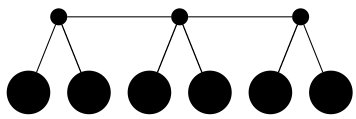
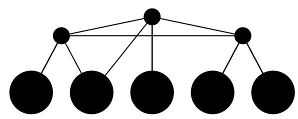
$\lambda_{\min}(m_9(t)) = -t - \sqrt{2}$. It means that the smallest eigenvalue of each matrix in $\mathscr{M}(t)$ is at most $-t - \sqrt{2}$. Therefore, the following result can be derived from Theorem 2.9.

Proposition 2.13 ([15, Proposition 3.3])**.** *Let $\mathfrak{h}$ be a Hoffman graph with special matrix in $\mathscr{M}(t)$. For fixed $\epsilon > 0$, there exists a positive integer $p := p(\epsilon, t)$ such that $\lambda_{\min}(G(\mathfrak{h}, p)) < -t - \sqrt{2} + \epsilon$.*

To simplify the narrative, we introduce nine Hoffman graphs in the following.

$\mathfrak{h}_{1,-2} =$  $\mathfrak{h}_{3,1} =$  $\mathfrak{h}_{3,-1} =$ 

$\mathfrak{h}_{4,-2} =$  $\mathfrak{h}_5 =$  $\mathfrak{h}_6 =$ 

$\mathfrak{h}_7 =$  $\mathfrak{h}_8^{(1)} =$  $\mathfrak{h}_8^{(2)} =$ 

We denote by $\mathfrak{H} := \{\mathfrak{h}_{1,-2}, \mathfrak{h}_{3,1}, \mathfrak{h}_{3,-1}, \mathfrak{h}_{4,-2}, \mathfrak{h}_5, \mathfrak{h}_6, \mathfrak{h}_7, \mathfrak{h}_8^{(1)}, \mathfrak{h}_8^{(2)}\}$.

Lemma 2.14. *Let $\mathfrak{h}$ be a Hoffman graph in which two slim vertices are adjacent if they share at least one fat common neighbor. If $\mathfrak{h}$ contains an induced Hoffman subgraph with special matrix in $\mathscr{M}(2)$, then it must contain a member of $\mathfrak{H}$ as an induced Hoffman subgraph.*

Proof. It can be confirmed by checking the possibilities of Hoffman graphs with special matrices in $\{m_{1,-2}(2),\ m_{2,-2}(2)\ m_{3,1}(2), m_{3,-1}(2),\ m_{3,-2}(2),\ m_{4,1}(2), m_{4,-1}(2),\ m_{4,-2}(2),\ m_{4,-3}(2), m_5(2), m_6(2), m_7(2), m_8(2), m_9(2)\}$. □

By calculation, we have the following proposition.

Proposition 2.15. *The following inequalities hold:*

$$\lambda_{\min}(G(\mathfrak{h}_{1,-2}, 7)) < -3, \quad \lambda_{\min}(G(\mathfrak{h}_{3,1}, 7)) < -3, \quad \lambda_{\min}(G(\mathfrak{h}_{3,-1}, 13)) < -3,$$
$$\lambda_{\min}(G(\mathfrak{h}_{4,-2}, 5)) < -3, \quad \lambda_{\min}(G(\mathfrak{h}_5, 11)) < -3, \quad \lambda_{\min}(G(\mathfrak{h}_6, 5)) < -3,$$
$$\lambda_{\min}(G(\mathfrak{h}_7, 15)) < -3, \quad \lambda_{\min}(G(\mathfrak{h}_8^{(1)}, 8)) < -3, \quad \lambda_{\min}(G(\mathfrak{h}_8^{(2)}, 11)) < -3.$$

Proof. By computer verification. □

Let $\mathfrak{h}_{(s)}$ the 2-fat Hoffman graph with s slim vertices and two fat vertices such that its slim graph is a complete graph, and let $\mathfrak{h}_{(2)}^{(s)}$ be the Hoffman graph consisting of two adjacent slim vertices, one of which is adjacent to exactly s fat vertices, while the other has no fat neighbors.

Proposition 2.16. (i) *If $p_1 := s(s-1) + 1$, then $\lambda_{\min}(G(\mathfrak{h}^{(s+1)}, p_1)) < -s$.*

(ii) *If* $p_2 := (s-1)(2s-1)+1$*, then* $\lambda_{\min}(G(\mathfrak{h}_{(s)}, p_2)) < -s$.

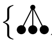
(iii) *If* $p_3 = (s+1)(s-1)^2+1$*, then* $\lambda_{\min}(G(\mathfrak{h}^{(s)}_{(2)}, p_3)) < -s$.

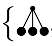
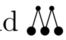
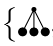
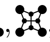
Proof. The graph $G(\mathfrak{h}^{(s+1)}, p_1)$ consists of a vertex u and $s+1$ pairwise disjoint cliques $C_1, \dots, C_{s+1}$. Note that $\{\{u\}, \bigcup_{i=1}^{s+1} V(C_i)\}$ is an equitable partition of $G(\mathfrak{h}^{(s+1)}, p_1)$ with quotient matrix $A_1 = \begin{pmatrix} 0 & (s+1)p_1 \\ 1 & p_1-1 \end{pmatrix}$ (for the notations of equitable partitions and quotient matrices, see [4, Chapter 9.3]). The smallest eigenvalue of A_1 is $\frac{1}{2}(p_1 - 1 - \sqrt{(p_1+1)^2+4sp_1})$, which is less than $-s$. Thus, $\lambda_{\min}(G(\mathfrak{h}^{(s+1)}, p_1)) < -s$ holds.

The graph $G(\mathfrak{h}_{(s)}, p_2)$ consists of s vertices $u_1, \dots, u_s$ and two disjoint cliques C_1, C_2. Note that $\{\{u_i\}_{i=1}^s, V(C_1) \cup V(C_2)\}$ is an equitable partition of $G(\mathfrak{h}_{(s)}, p_2)$ with quotient matrix $A_2 = \begin{pmatrix} s-1 & 2p_2 \\ s & p_2-1 \end{pmatrix}$. The smallest eigenvalue of A_1 is $\frac{1}{2}(s + p_2 - 2 - \sqrt{(s+p_2)^2+4p_2 s})$, which is less than $-s$. Thus, $\lambda_{\min}(G(\mathfrak{h}_{(s)}, p_2)) < -s$ holds.

The graph $G(\mathfrak{h}^{(s)}_{(2)}, p_3)$ consists of two vertices u, v and s pairwise disjoint cliques $C_1, \dots, C_s$. Note that $\{\{u\}, \{v\}, \bigcup_{i=1}^{s} V(C_i)\}$ is an equitable partition of $G(\mathfrak{h}^{(s)}_{(2)}, p_3)$ with quotient matrix $A_3 = \begin{pmatrix} 0 & 1 & sp_3 \\ 1 & 0 & 0 \\ 1 & 0 & p_3-1 \end{pmatrix}$. It follows that $\det(A_3 + sI) = (s+1)(s-1)^2 - p_3 < 0$. Thus, $\lambda_{\min}(G(\mathfrak{h}^{(s+1)}, p_1)) < -s$ holds. □

Let t be a positive integer. We denote by $\mathfrak{G}(t)$ the family of pairwise non-isomorphic indecomposable t-fat Hoffman graphs whose special matrix is either $\begin{pmatrix} J_{r_1} & -J_{r_1 \times r_2} \\ -J_{r_2 \times r_1} & J_{r_2} \end{pmatrix} - (t+1)I$ or $(-t-1)$, where $r_1, r_2 \in [t]$. Note that $\mathfrak{G}(t)$ is a finite family of Hoffman graphs.

Lemma 2.17. *A* $\mathfrak{G}(2)$*-line Hoffman graph is a* $\{$, , $\}$*-line Hoffman graph.*

Proof. Note that $\mathfrak{G}(2) = \{$, , , , $\}$. As  and  both are induced Hoffman subgraphs of . Thus, a $\mathfrak{G}(2)$-line Hoffman graph is a $\{$, , $\}$-line Hoffman graph. □

Koolen et al.[15, Theorem 3.7] showed the following result, which relates $\mathscr{M}(t)$ and $\mathfrak{G}(t)$. Two matrices B_1 and B_2 are *equivalent* if there exists a permutation matrix P such that $P^T B_1 P = B_2$.

Theorem 2.18 ([15, Theorem 3.7])**.** *Let* t *be a positive integer and* $\mathfrak{h}$ *a* t*-fat Hoffman graph. If the special matrix* $S(\mathfrak{h})$ *does not contain any principal submatrix equivalent to an element of* $\mathscr{M}(t)$*, then* $\mathfrak{h}$ *is a* t*-fat* $\mathfrak{G}(t)$*-line Hoffman graph.*

3 Associated Hoffman graphs of μ-bounded graphs

In this section, we first give a simplified definition for the associated Hoffman graph of a μ-bounded graph. For a more general definition of associated Hoffman graphs by using quasi-cliques, we refer readers to [9]. For μ-bounded graphs, these two definitions are equivalent if the order of each maximal clique in the graph is large enough.

Definition 3.1. *Let $q \geq 2$ be an integer and G a μ-bounded graph. Let $\mathscr{C}(q) := \{C_1, \dots, C_r\}$ be the set of all maximal cliques in G with at least q vertices. The* associated Hoffman graph *$\mathfrak{g} = \mathfrak{g}(G, q)$ is the Hoffman graph satisfying the following conditions:*

(i) *$V_{\text{slim}}(\mathfrak{g}) = V(G)$ and $V_{\text{fat}}(\mathfrak{g}) = \{f_1, \dots, f_r\}$,*

(ii) *the slim graph of $\mathfrak{g}$ is equal to G,*

(iii) *f_i is adjacent precisely to all vertices in C_i for $i \in [r]$.*

Note that if two distinct slim vertices u and v in an associated Hoffman graph have a common fat neighbor, then u is adjacent to v. For a μ-bounded graph G with parameter c, the following proposition characterizes the adjacency matrix of G if $\mathfrak{g}(G, q)$ is a 2-fat $\{ , \}$-line Hoffman graph, and thus $\lambda_{\min}(G) \geq -3$.

We denote by $\mathbf{j}$ the all one vector. For a vector $\mathbf{v}$, we denote by $\text{supp}(\mathbf{v}) := \{r \mid (\mathbf{v})_r \neq 0\}$ its support. For two vectors $\mathbf{a} = (a_1, \dots, a_m)^T$ and $\mathbf{b} = (b_1, \dots, b_n)^T$, we denote by $\mathbf{a} \oplus \mathbf{b}$ the vector $(a_1, \dots, a_m, b_1, \dots, b_n)^T$.

Proposition 3.2. *Let c and q be positive integers. Let G be a μ-bounded graph with parameter c. If the associated graph $\mathfrak{g}(G, q)$ is a 2-fat $\{ , \}$-line Hoffman graph, then there exists a $(0, \pm 1)$-matrix N whose columns are indexed by the vertex set $V(G)$ and satisfying the following properties:*

(i) *$A(G) + 3I = N^T N$ and $\lambda_{\min}(G) \geq -3$,*

(ii) *$N_v^T N_v = 3$ and $\mathbf{j}^T N_v \in \{1, 3\}$ for each vertex $v \in V(G)$,*

(iii) *if $\mathbf{j}^T N_v = 1$, then there is a vertex $u \in V(G)$ such that $N_v^T N_u = 1$ and $\text{supp}(N_v) = \text{supp}(N_u)$.*

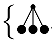
Proof. By definition 2.12, there is a Hoffman graph $\mathfrak{h} = \uplus_{i=1}^{t} \mathfrak{h}^i$ containing $\mathfrak{g}(G, q)$ as an induced Hoffman subgraph such that $V_{\text{slim}}(\mathfrak{h}) = V_{\text{slim}}(\mathfrak{g}(G, q)) = V(G)$, where $\mathfrak{h}^i$ is isomorphic to a member of $\{ , , \}$ for $i \in [t]$. By Definition 2.10, $S(\mathfrak{h}) = A(G) - D^T D$ is a block diagonal matrix with blocks in $\{S( ), S( ), S( )\}$, where $S( ) = (-3)$, $S( ) = (-2)$, and $S( ) = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$.

For $v \in V(G)$, let D_v be the column of D indexed by the vertex v. Let $\mathbf{e}_v$ be the vector indexed by the vertices of $V(G)$ such that the v-entry is one and the others are zero.

Let $\mathfrak{h}' \in \{\mathfrak{h}^i\}_{i=1}^{t}$. If $\mathfrak{h}' =$ and $V_{\text{slim}}(\mathfrak{h}') = \{v\}$, then let $N_v = D_v$. If $\mathfrak{h}' =$ and $V_{\text{slim}}(\mathfrak{h}') = \{v\}$, then let $N_v = D_v \oplus \mathbf{e}_v$. If $\mathfrak{h}' =$ and $V_{\text{slim}}(\mathfrak{h}') = \{u, v\}$, then let $N_v = D_v \oplus \mathbf{e}_v$ and

$N_u = D_u \oplus (-\mathbf{e}_v)$. Let N be the matrix such that the column indexed by v is N_v for $v \in V(G)$. Therefore, N is a $(0, \pm 1)$-matrix satisfying (ii) and (iii).

Let u and v be two distinct vertices in $V(G)$. If u, v are not contained in the same member of $\{\mathfrak{h}^i\}_{i=1}^t$, then $(N_u)^T N_v = (D_u)^T D_v$ and $(S(\mathfrak{h}))_{uv} = 0$ by Lemma 2.11. It follows that $(N_u)^T N_v = (D_u)^T D_v = (A(G) - S(\mathfrak{h}))_{uv} = (A(G))_{uv}$. If u, v are contained in the member $\mathfrak{h}'$ of $\{\mathfrak{h}^i\}_{i=1}^t$, then $\mathfrak{h}' =$ . It follows that $(N_u)^T N_v = (D_u)^T D_v - 1$ and $(S(\mathfrak{h}))_{uv} = -1$. Furthermore, $(N_u)^T N_v = (D_u)^T D_v - 1 = (A(G) - S(\mathfrak{h}))_{uv} - 1 = (A(G))_{uv}$. Therefore, $A(G) + 3I = N^T N$, and thus $\lambda_{\min}(G) \geq -3$. $\square$

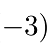
For a μ-bounded graph G with parameter c, the following proposition characterizes the structure of G if $\mathfrak{g}(G, q)$ is a 2-fat $\{$, $\}$-line Hoffman graph.

Proposition 3.3. *Let c and q be positive integers. Let G be a μ-bounded graph with parameter c. If the associated graph $\mathfrak{g}(G, q)$ is a 2-fat $\{$, $\}$-line Hoffman graph, then there exists a set of cliques $\mathfrak{C}$ in G satisfying the following:*

(i) $E(G) = \bigcup_{C \in \mathfrak{C}} E(C)$,

(ii) *for every vertex $u \in V(G)$, there are at most three cliques in $\mathfrak{C}$ containing u, where two of them are maximal cliques with order at least q,*

(iii) *if there are distinct cliques $C, C' \in \mathfrak{C}$ such that $V(C) \cap V(C') \supset \{u, v\}$, then C and C' are the only two cliques in $\mathfrak{C}$ containing u (resp. v) and $V(C) \cap V(C') = \{u, v\}$.*

Proof. By definition 2.12, there is a Hoffman graph $\mathfrak{h} = \uplus_{i=1}^t \mathfrak{h}^i$ containing $\mathfrak{g}(G, q)$ as an induced Hoffman subgraph such that $V_{\text{slim}}(\mathfrak{h}) = V_{\text{slim}}(\mathfrak{g}(G, q)) = V(G)$, where $\mathfrak{h}^i$ is isomorphic to a member of $\{$, , $\}$ for $i \in [t]$. By Definition 2.10, $S(\mathfrak{h}) = A(G) - D^T D$ is a block diagonal matrix with blocks in $\{S($$)$, $S($$)$, $S($$)\}$, where $S($$) = (-3)$, $S($$) = (-2)$, and $S($$) = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$.

For a fat vertex f in $\mathfrak{h}$, let $V_f = \{u \in V(G) \mid (D)_{fu} = 1\}$. Let C_f be the subgraph of G induced by V_f for $f \in V_{\text{fat}}(\mathfrak{h})$. Assume that there are two distinct non-adjacent vertices u_1, u_2 in C_f, then u_1, u_2 have one common fat neighbor. Hence, u_1, u_2 must be contained in a Hoffman graph $\mathfrak{h}' \in \{\mathfrak{h}^i\}_{i=1}^t$ by Lemma 2.11 (iv). It follows that $\mathfrak{h}' =$ , which contradicts that u_1, u_2 are non-adjacent. Therefore, C_f is a clique in G for $f \in V_{\text{fat}}(\mathfrak{h})$.

Let $\mathfrak{C} := \{C_f \mid f \in V_{\text{fat}}(\mathfrak{h})\}$. Assume that there is an edge uv not contained in any clique in $\mathfrak{C}$, then $(D_u)^T D_v = 0$. It follows that $(S(\mathfrak{h}))_{uv} = (S(\mathfrak{h}))_{uv} + (D_u)^T D_v = (A(G))_{uv} = 1$. However, every off-diagonal entry of $S(\mathfrak{h})$ is not equal to 1. Hence, (i) holds.

As $\mathfrak{g}(G, q)$ is 2-fat, each vertex in $V(G)$ is contained in at least two maximal cliques with order at least q by Definition 3.1. For $v \in V(G)$, the Hoffman graph in $\{\mathfrak{h}^i\}_{i=1}^t$ containing v is isomorphic to a member of $\{$, , $\}$. Therefore, each vertex is contained in at most three cliques in $\mathfrak{C}$, and (ii) holds.

If there are two cliques $C_{f_1}, C_{f_2} \in \mathfrak{C}$ such that $V_{f_1} \cap V_{f_2}$ contains two distinct vertices u, v, then u and v have two common fat neighbors. Hence, $V_{f_1} \cap V_{f_2}$ must be contained in a Hoffman graph $\mathfrak{h}'' \in \{\mathfrak{h}_i\}_{i=1}^{t}$ by Lemma 2.11 (iv). It follows that $\mathfrak{h}'' =$. This implies that C_{f_1}, C_{f_2} are the only two cliques in $\mathfrak{C}$ containing u (resp. v) and $V_{f_1} \cap V_{f_2} = \{u, v\}$. Therefore, (iii) holds. $\square$

As an application of Proposition 3.3, the following lemma characterizes local graphs of G.

Corollary 3.4. *Let c and q be positive integers. Let G be a μ-bounded graph with parameter c. If the associated graph $\mathfrak{g}(G, q)$ is a 2-fat $\{$, $\}$-line Hoffman graph, then $c \leq 18$. In particular, if G is regular, then $c \leq 12$.*

Proof. Let $\mathfrak{C}$ be a set of cliques satisfying (i)-(iii) of Proposition 3.3. Let $\mathfrak{C}(v) := \{C \in \mathfrak{C} \mid v \in C\}$ for $v \in V(G)$. Let u_1 and u_2 be two non-adjacent vertices in G.

Proposition 3.3 (ii) shows that $|\mathfrak{C}(u_i)| \leq 3$ for $i = 1, 2$, and $|V(A) \cap V(B)| \leq 2$ for $A \in \mathfrak{C}(u_1)$ and $B \in \mathfrak{C}(u_2)$. Hence, $|N(u_1) \cap N(u_2)| \leq \sum_{A \in \mathfrak{C}(u_1)} \sum_{B \in \mathfrak{C}(u_2)} |V(A) \cap V(B)| \leq 2|\mathfrak{C}(u_1)||\mathfrak{C}(u_2)| = 2 \times 3 \times 3 = 18$.

Assume that G is k-regular, and $|N(u_1) \cap N(u_2)| \geq 13$. It follows that $|\mathfrak{C}(u_1)| = |\mathfrak{C}(u_2)| = 3$, otherwise $|N(u_1) \cap N(u_2)| \leq 2 \times 2 \times 3 = 12$. Let $\mathfrak{C}(u_1) = \{A_1, A_2, A_3\}$, $\mathfrak{C}(u_2) = \{B_1, B_2, B_3\}$, $x_i := |V(A_i)| - 1$, and $y_i := |V(B_i)| - 1$ for $i = 1, 2, 3$. Note that $x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = k$. As $|N(u_1) \cap N(u_2)| \geq 13$, there are at least 4 pairs of (i, j) such that $|V(A_i) \cap V(B_j)| = 2$ and $x_i + y_j = k + 1$. It can be directly confirmed that there does not exist a solution $(x_1, x_2, x_3, y_1, y_2, y_3)$ satisfying these equations. $\square$

Lemma 3.5. *Let c and q be positive integers. Let G be a μ-bounded graph with parameter c. If the associated graph $\mathfrak{g}(G, q)$ is a 2-fat $\{$, $\}$-line Hoffman graph, then there exists a vertex u in G such that $|N(u) \cap N(v)| \leq 9$ for each vertex v non-adjacent to u.*

Proof. Let $\mathfrak{C}$ be a set of cliques satisfying (i)-(iii) of Proposition 3.3. Let $\mathfrak{C}(w) := \{C \in \mathfrak{C} \mid w \in C\}$ for $w \in V(G)$.

If there are two cliques C_1 and C_2 in $\mathfrak{C}$ such that $V(C_1) \cap V(C_2) = \{u, u'\}$, then u is the desired vertex. We will show that $|N(u) \cap N(v)| \leq 9$ for each vertex v non-adjacent to u. Note that $\mathfrak{C}(u) = \{C_1, C_2\}$ by Proposition 3.3 (iii). Assume that there exists a vertex $v \notin N(u)$ such that $|N(u) \cap N(v)| \geq 10$. It follows that $|\mathfrak{C}(v)| = 3$, otherwise $|N(u) \cap N(v)| \leq 2 \times 2 \times 2 = 8$. Let A_1, A_2, A_3 be the cliques in $\mathfrak{C}(v)$. Let $x_i := |V(A_i)| - 1$ and $y_j := |V(C_j)| - 1$ for $i = 1, 2, 3$ and $j = 1, 2$. Note that $x_1 + x_2 + x_3 = k = y_1 + y_2 - 1$. As $|N(u) \cap N(v)| \geq 10$, there are at least 4 pairs of (i,j) such that $|V(A_i)\cap V(C_j)|=2$ and $x_i+y_j=k+1$. It can be directly confirmed that there does not exist a solution (x_1,x_2,x_3,y_1,y_2) satisfying these equations.

If each pair of distinct cliques in $\mathfrak{C}$ intersects in at most one vertex, then $c\le 3\times 3\times 1=9$ by Proposition 3.3 (ii). In other words, for each vertex w in G, it shares at most 9 common neighbors with every vertex non-adjacent to w. This shows the existence of u. $\square$

The following result states that for a fixed smallest eigenvalue, each vertex outside a big enough clique has either a large number of neighbors or a large number of non-neighbors in that clique.

Lemma 3.6 ([24, Lemma 1.2])**.** *Let $\lambda\ge 1$ be an integer and G a graph with $\lambda_{\min}(G)\ge-\lambda$. Let C be a clique in G with order ω. If $\omega\ge n_1(\lambda):=\lambda^4-2\lambda^3+3\lambda^2-3\lambda+3$, then each vertex outside C has either at most $\lambda(\lambda-1)$ neighbors in C or at least $\omega-(\lambda-1)^2$ neighbors in C.*

The following result is a version for μ-bounded graphs of [9, Proposition 4.1], which shows the relation between graphs and associated Hoffman graphs.

Proposition 3.7. *Let ϕ,σ and p be non-negative integers. Let λ,c be positive integers and $\tilde{c}:=\min\{c,\lambda(\lambda-1)\}$. Let $n_1(\lambda):=\lambda^4-2\lambda^3+3\lambda^2-3\lambda+3$. There exists a positive integer $n_2(\phi,\sigma,p,\tilde{c}):=\tilde{c}(\sigma-1)+\tilde{c}(p+1)(\phi-1)+p+1$ such that for each integer $q\ge\max\{n_1(\lambda),n_2(\phi,\sigma,p,\tilde{c}),(\lambda-1)^2+c+1\}$, every Hoffman graph $\mathfrak{h}$ with ϕ fat vertices and σ slim vertices and every graph G with $\lambda_{\min}(G)\ge-\lambda$, the graph $G(\mathfrak{h},p)$ is an induced subgraph of G if the graph G satisfies the following conditions:*

(i) *the graph G is a μ-bounded graph with parameter c,*

(ii) *the associated Hoffman graph $\mathfrak{g}(G,q)$ contains $\mathfrak{h}$ as an induced Hoffman subgraph.*

Proof. For $\phi=0$, the proposition holds. Assume that the proposition holds for $\phi-1$, we will show that the proposition holds for $\phi\ge 1$.

Let $V_{\text{fat}}(\mathfrak{h})=\{f_i\}_{i=1}^{\phi}$ and $V_{\text{slim}}(\mathfrak{h})=\{s_i\}_{i=1}^{\sigma}$. As $\mathfrak{h}$ is an induced Hoffman subgraph of $\mathfrak{g}(G,q)$, there is a set $\{C_i\}_{i=1}^{\phi}$ of the maximal cliques with order at least q in G corresponding to $\{f_i\}_{i=1}^{\phi}$. Additionally, the slim graph of $\mathfrak{h}$ is an induced subgraph of G as the slim graph of $\mathfrak{g}(G,q)$ is G.

Let $\mathfrak{h}':=\mathfrak{h}-f_\phi$ be the Hoffman graph obtained by deleting the fat vertex f_ϕ from $\mathfrak{h}$. As $n_2(\phi,\sigma,p,\tilde{c})\ge n_2(\phi-1,\sigma,p,\tilde{c})$, by induction, $G(\mathfrak{h}',p)$ is an induced subgraph of G. In other words, there exist cliques $D_1,\dots,D_{\phi-1}$ such that

(a_1) for $i\in[\phi-1]$, $V(D_i)\cap V_{\text{slim}}(\mathfrak{h})=\emptyset$, $V(D_i)\subset V(C_i)$, and $|V(D_i)|\ge q$,

(a_2) for distinct $i,j\in[\phi-1]$, $V(D_i)\cap V(D_j)=\emptyset$, and no edge in G exists between D_i and D_j,

(a_3) for $\ell\in[\sigma]$ and $i\in[\phi-1]$, the slim vertex s_ℓ is adjacent to all vertices of D_i if s_ℓ is adjacent to the fat vertex f_i, and s_ℓ has no neighbors in D_i if s_ℓ is not adjacent to f_i.

As G is a μ-bounded graph with parameter c and $|V(C_i)| \geq q \geq \max\{n_1(\lambda), (\lambda-1)^2+c+1\}$ for $i \in [\phi]$. Lemma 3.6 implies that $|N_G(x) \cap V(C_i)| \leq \tilde{c}$ for every vertex x outside C_i. Thus, $|V(C_i) \cap V(C_j)| \leq \tilde{c}$ for distinct $i, j \in [\phi]$. By Definition 3.1, $s_\ell \in V(C_i)$ if s_ℓ is adjacent to f_i, for $\ell \in [\sigma]$ and $i \in [\phi]$. Hence, the slim vertex s_ℓ has at most $\tilde{c}$ neighbors in C_i if s_ℓ is not adjacent to f_i.

Let $A_\phi := \{s \in V_{\text{slim}}(\mathfrak{h}) \mid s \text{ is adjacent to } f_\phi\}$ and $A^*_\phi := V_{\text{slim}}(\mathfrak{h}) - A_\phi$. Note that $|A_\phi| + \tilde{c}|A^*_\phi| \leq 1 + \tilde{c}(\sigma-1)$ as $|A_\phi| + |A^*_\phi| = \sigma$ and $\tilde{c} \geq 1$. It follows that $|V(C_\phi) - A_\phi - \bigcup_{x \in A^*_\phi} N_G(x) - \bigcup_{i \neq \phi} \bigcup_{y \in D_i} N_G(y) - \bigcup_{i \neq \phi} V(C_i)| \geq q - (1 + \tilde{c}(\sigma-1)) - \tilde{c}p(\phi-1) - \tilde{c}(\phi-1) \geq p$ holds. Therefore, there is a clique D_ϕ satisfying

- (b_1) $|V(D_\phi)| = p$ and $V(D_\phi) \subset V(C_\phi) - A_\phi - \bigcup_{x \in A^*_\phi} N_G(x) - \bigcup_{i \neq \phi} \bigcup_{y \in D_i} N_G(y) - \bigcup_{i \neq \phi} V(C_i)$,
- (b_2) for $i \in [\phi-1]$, $V(D_\phi) \cap V_{\text{slim}}(\mathfrak{h}) = \emptyset$, $V(D_i) \cap V(D_\phi) = \emptyset$, and no edge in G exists between D_i and D_ϕ,
- (b_3) for $\ell \in [\sigma]$, the slim vertex s_ℓ is adjacent to all vertices of D_ϕ if s_ℓ is adjacent to the fat vertex f_ϕ, and s_ℓ has no neighbors in D_ϕ if s_ℓ is not adjacent to f_ϕ.

It follows that the subgraph in G induced by $(\bigcup_{i=1}^{\phi} V(D_i)) \cup \{s_1, \ldots, s_\phi\}$ is isomorphic to $G(\mathfrak{h}, p)$. □

Theorem 3.8. *Let $c, \tilde{c}$ and q be positive integers such that $\tilde{c} = \min\{c, 6\}$ and $q \geq \max\{c+5, 50\tilde{c}+16\}$. If G is a μ-bounded graph with parameter c and $\lambda_{\min}(G) \geq -3$, then the associated Hoffman graph $\mathfrak{g}(G, q)$ does not contain an induced Hoffman subgraph with special matrix in $\mathscr{M}(2)$.*

Proof. By Definition 3.1, two distinct slim vertices u and v in $\mathfrak{g}(G, q)$ are adjacent if they have at least one common fat neighbor. Assume that $\mathfrak{g}(G, q)$ contains an induced Hoffman subgraph with special matrix in $\mathscr{M}(2)$. Lemma 2.14 shows that $\mathfrak{g}(G, q)$ contains an induced Hoffman subgraph in $\mathfrak{H} = \{\mathfrak{h}_{1,-2}, \mathfrak{h}_{3,1}, \mathfrak{h}_{3,-1}, \mathfrak{h}_{4,-2}, \mathfrak{h}_5, \mathfrak{h}_6, \mathfrak{h}_7, \mathfrak{h}_8^{(1)}, \mathfrak{h}_8^{(2)}\}$.

Since $\lambda_{\min}(G) \geq -3$, by Proposition 2.15, G does not contain a member of $\{G(\mathfrak{h}_{1,-2}, 7)$, $G(\mathfrak{h}_{3,1}, 7)$, $G(\mathfrak{h}_{3,-1}, 13)$, $G(\mathfrak{h}_{4,-2}, 5)$, $G(\mathfrak{h}_5, 11)$, $G(\mathfrak{h}_6, 5)$, $G(\mathfrak{h}_7, 15)$, $G(\mathfrak{h}_8^{(1)}, 8)$, $G(\mathfrak{h}_8^{(2)}, 11)\}$ as an induced subgraph. Note that $q \geq \max\{c+5, 50\tilde{c}+16\} \geq \max\{n_2(4,1,7,\tilde{c}), n_2(5,2,7,\tilde{c}), n_2(3,2,13,\tilde{c})$, $n_2(3,2,5,\tilde{c})$, $n_2(2,3,11,\tilde{c})$, $n_2(4,3,5,\tilde{c})$, $n_2(4,3,15,\tilde{c})$, $n_2(6,3,8,\tilde{c})$, $n_2(5,3,11,\tilde{c})\}$, and $50\tilde{c}+16 \geq n_1(3) = 48$. It follows that $\mathfrak{g}(G, q)$ does not contain an induced subgraph isomorphic to a member of $\mathfrak{H}$ by Proposition 3.7, which contradicts with Lemma 2.14. □

4 Main theorems

In this section, we complete the proof of Theorem 1.6. The following key lemma is inspired by the Bose–Laskar method, which was first introduced by Bose and Laskar in [1] in 1967. Using this approach, they established fundamental relationships between claws and cliques in edge-regular graphs, leading to a characterization of tetrahedral graphs. Subsequently, Huang [7] applied this method to characterize bilinear forms graphs. Metsch later refined the Bose–Laskar method and applied it to improve Bruck's completion theorem [16]. Based on Metsch's refinement, further characterizations were obtained, including those of Grassmann graphs [10, 17] and bilinear forms graphs [18]. The Bose–Laskar method is also closely related to the claw bound. This bound was first proved by Metsch [16, Lemma 1.1], and was later rediscovered by Godsil [5, Lemma 2.3] and by Koolen and Park [13, Lemma 2]. The claw bound plays an important role in the characterization of strongly regular graphs; see, for example, [11, 19].

Lemma 4.1. *Let c and r be positive integers. If G is a μ-bounded graph with parameter c and $\lambda_{\min}(G) \geq -\lambda$, then each vertex $x \in V(G)$ lies in a maximal clique $C_1(x)$ of order at least $\frac{d(x)-\binom{\lfloor\lambda^2\rfloor}{2}(c-1)}{\lfloor\lambda^2\rfloor}+1$. Moreover, if every maximal clique containing x has order at most $d(x)-r$, then x lies in another distinct maximal clique $C_2(x)$ of order at least $\frac{r-\binom{\lfloor\lambda^2\rfloor}{2}(c-1)+1}{\lfloor\lambda^2\rfloor-1}+1$.*

Proof. Since $\lambda_{\min}(K_{1,\lfloor\lambda^2+1\rfloor}) = -\sqrt{\lfloor\lambda^2+1\rfloor} < -\lambda$, G is induced $K_{1,\lfloor\lambda^2+1\rfloor}$-free by interlacing theorem. Hence, the maximum order of independent sets in $N(x)$ is at most $\lfloor\lambda^2\rfloor$ for any vertex $x \in V(G)$. Let $I = \{v_1, v_2, ..., v_s\}$ be a maximum independent set in $N(x)$, where $s \leq \lfloor\lambda^2\rfloor$.

Let $W = \{y \in N(x) \mid |N(y) \cap I| \geq 2\}$. As each pair of two distinct non-adjacent vertices has at most c common neighbors, $|W| \leq \binom{s}{2}(c-1) \leq \binom{\lfloor\lambda^2\rfloor}{2}(c-1)$.

Let $P_i := \{y \in N(x) \mid v_i \in N(y)\} \cup \{v_i\} - W$ for $i \in [s]$. It follows that $P_i \cap P_j = \emptyset$ for $i \neq j$. If there are two distinct non-adjacent vertices $y_1, y_2 \in P_i$, then $(I\backslash\{v_i\}) \cup \{y_1, y_2\}$ is an independent set of order $s+1 > |I| = s$. Hence, the subgraph induced by P_i is a clique containing v_i for $i \in [s]$.

As the disjoint union $W \cup P_1 \cdots \cup P_s$ is exactly the set $N(x)$, $|W| + \sum_{i=1}^{s} |P_i| = d(x)$ holds. Hence, $\sum_{i=1}^{s} |P_i| \geq d(x) - \binom{\lfloor\lambda^2\rfloor}{2}(c-1)$. Assume that $|P_1| \geq |P_2| \geq \cdots \geq |P_s|$, then the subgraph induced by $P_1 \cup \{x\}$ is a clique of order at least $\frac{d(x)-\binom{\lfloor\lambda^2\rfloor}{2}(c-1)}{s}+1 \geq \frac{d(x)-\binom{\lfloor\lambda^2\rfloor}{2}(c-1)}{\lfloor\lambda^2\rfloor}+1$.

If every maximal clique containing x has order at most $d(x)-r$, then $|P_1| \leq d(x)-r-1$ and the subgraph induced by $P_2 \cup \{x\}$ is a clique of order at least $\frac{d(x)-\binom{\lfloor\lambda^2\rfloor}{2}(c-1)-(d(x)-r-1)}{s-1}+1 \geq \frac{r-\binom{\lfloor\lambda^2\rfloor}{2}(c-1)+1}{\lfloor\lambda^2\rfloor-1}+1$. □

Without using Ramsey theory, the above lemma helps us to find large cliques in μ-bounded graphs which are necessary for Hoffman theory.

Corollary 4.2. *Let c and q be positive integers. Let G be a μ-bounded graph with parameter c and $\lambda_{\min}(G) \geq -3$. If every clique C in G has order at most $\min_{x\in V(C)} d(x) - 36c + 45 - 8q$, then the associated Hoffman graph $\mathfrak{g}(G, q)$ is 2-fat.*

Proof. By Lemma 4.1, for each vertex v, there are two distinct maximal cliques $C_1(v)$ and $C_2(v)$ containing v such that $|V(C_1(v))| \geq \frac{d(v)-\binom{9}{2}(c-1)}{9} + 1 \geq \frac{d(v)-36c+45}{9}$ and $|V(C_2(v))| \geq \frac{1}{8}(d(v) - \min_{x\in V(C_2(v))} d(x) + 36c - 45 + 8q - \binom{9}{2}(c-1) + 1) + 1 \geq \frac{1}{8}(36c - 45 + 8q - \binom{9}{2}(c-1) + 1) + 1 = q$. Note that $d(v) \geq 9q + 36c - 45$ as $q \leq |V(C_2(v))| \leq \min_{x\in V(C_2(v))} d(x) - 36c + 45 - 8q$. It follows that $|V(C_1(v))| \geq \frac{d(v)-36c+45}{9} \geq q$. In other words, each vertex is contained in two maximal cliques with order at least q, and thus $\mathfrak{g}(G, q)$ is 2-fat by Definition 3.1. $\square$

Now, we are in the position to prove Theorem 1.6.

Proof of Theorem 1.6. Let $q := \max\{c + 5, 50\tilde{c} + 16\}$. By the condition (ii), every clique C in G has order at most $\min_{x\in V(C)} d(x) - 36c + 45 - 8q$. Thus, the associated Hoffman graph $\mathfrak{g}(G, q)$ is 2-fat by Corollary 4.2. By Theorem 3.8, $\mathfrak{g}(G, q)$ does not contain an induced Hoffman subgraph whose special matrix is a member of $\mathscr{M}(2)$. So, the associated Hoffman graph $\mathfrak{g}(G, q)$ is a 2-fat $\mathfrak{G}(2)$-line Hoffman graph by Theorem 2.18.

Notice that a $\mathfrak{G}(2)$-line Hoffman graph is a $\{\ ,\ ,\ \}$-line Hoffman graph by Lemma 2.17. This means that, by Definition 2.12, $\mathfrak{g}(G, q)$ is a 2-fat induced Hoffman subgraph of a Hoffman graph $\mathfrak{h} = \uplus_{i=1}^{v} \mathfrak{h}^i$, where $\mathfrak{h}^i$ is isomorphic to a 2-fat induced Hoffman subgraph of a Hoffman graph in $\{\ ,\ ,\ \}$ for $i \in [v]$.

Assume that $\mathfrak{h}^i$ is isomorphic to a 2-fat induced Hoffman subgraph of . It follows that $\mathfrak{h}^i$ belongs to the set $\mathscr{S} := \{\ ,\ ,\ ,\ ,\ \}$. In the following, we will show that $\mathfrak{h}^i$ is isomorphic to either or . It suffices to prove that each pair of slim vertices in $\mathfrak{h}^i$ is adjacent.

If there are two distinct non-adjacent slim vertices s_1, s_2 in $\mathfrak{h}^i$, then they have a common fat neighbor f in $\mathfrak{h}_i$, as $\mathfrak{h}_i \in \mathscr{S}$. For $j = 1, 2$, since s_j has exactly two fat neighbors in $\mathfrak{h}^i$, we may let f_j denote the other one. By Lemma 2.11 (iii), the set $N^{\text{fat}}_{\mathfrak{h}}(s_j)$ of fat neighbors of s_j in $\mathfrak{h}$ is contained in $N^{\text{fat}}_{\mathfrak{h}^i}(s_j)$ for $j = 1, 2$, thus $N^{\text{fat}}_{\mathfrak{h}}(s_j) = N^{\text{fat}}_{\mathfrak{h}^i}(s_j)$. By Definition 2.12, $\mathfrak{g}(G, q)$ is a 2-fat induced Hoffman subgraph of $\mathfrak{h}$. It follows that $2 \leq |N^{\text{fat}}_{\mathfrak{g}(G,q)}(s_j)| \leq |N^{\text{fat}}_{\mathfrak{h}}(s_j)| = |N^{\text{fat}}_{\mathfrak{h}^i}(s_j)| = 2$, and thus $N^{\text{fat}}_{\mathfrak{g}(G,q)}(s_j) = N^{\text{fat}}_{\mathfrak{h}^i}(s_j) = \{f, f_j\}$ for $j = 1, 2$. Therefore the two non-adjacent slim vertices s_1 and s_2 have a common fat neighbor f in $\mathfrak{g}(G, q)$. However, this contradicts Definition 3.1 (iii), which states that the set of slim neighbors of a fat vertex in $\mathfrak{g}(G, q)$ forms a clique in G. Thus, $\mathfrak{h}^i$ is isomorphic to either or .

Note that is an induced Hoffman subgraph of $=$ $\uplus$. It follows that both and are 2-fat -line Hoffman graphs. This shows that $\mathfrak{g}(G, q)$ is a 2-fat $\{\ ,\ \}$-line Hoffman graphs. $\square$

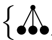
As we mentioned in Section 2.4, the smallest eigenvalue of each matrix in $\mathscr{M}(2)$ is at most $-2 - \sqrt{2}$. We give the following result without a proof, as it can be proved by Proposition 2.13, Proposition 3.7, and a similar discussion of the proof for Theorem 1.6.

Theorem 4.3. *Let c be a positive integer. For $\epsilon > 0$, there exists a positive integer $K := K(\epsilon, c)$ such that if a graph G satisfies the following conditions:*

(i) *G is a μ-bounded graph with parameter c,*

(ii) *every clique C has order at most $\min_{x\in V(C)} d(x) - K$,*

(iii) *$\lambda_{\min}(G) \geq -2-\sqrt{2}+\epsilon$,*

then the associated Hoffman graph $\mathfrak{g}(G, q)$ is a 2-fat $\{$, $\}$ *-line Hoffman graph and $\lambda_{\min}(G) \geq -3$.*

Remark 4.4. *The parameter $K = K(\epsilon, c)$ can be bounded by $\frac{500}{\epsilon} + 55c$ when ϵ tends to zero.*

The following proposition generalizes Proposition 3.3.

Proposition 4.5. *Let $\lambda \geq 2$ and c be positive integers and $\tilde{c} := \min\{c, \lambda(\lambda-1)\}$. Let $n_1(\lambda) := \lambda^4-2\lambda^3+3\lambda^2-3\lambda+3$ and $n_2(\phi,\sigma,p,\tilde{c}) := \tilde{c}(\sigma-1)+\tilde{c}(p+1)(\phi-1)+p+1$. Let $q \geq \max\{n_1(\lambda), n_2(\lambda+1, 1, \lambda(\lambda-1)+1, \tilde{c}), n_2(\lambda, 2, (\lambda+1)(\lambda-1)^2+1, \tilde{c}), (\lambda-1)^2+c+1\}$. Let $R(c,\lambda,q) := \lambda^2(q-1)+(c-1)\binom{\lambda^2}{2}$. If G is a μ-bounded graph with parameter c and $\lambda_{\min}(G) \geq -\lambda$, then there exists a set of maximal cliques $\mathfrak{P} = \{P_1, \ldots, P_s\}$ in G satisfying the following properties:*

(i) *$|N(x) - \bigcup_{i=1}^{s}\{y \mid x, y \in V(P_i)\}| < R(c,\lambda,q)$ for $x \in V(G)$,*

(ii) *$|\{i \mid x \in V(P_i)\}| \leq \lambda$ for $x \in V(G)$, and if the equality holds for x, then $N(x) = \bigcup_{i=1}^{s}\{y \mid x, y \in V(P_i)\} - \{x\}$,*

(iii) *$|V(P_i) \cap V(P_j)| \leq \lambda - 1$ for $i \neq j$,*

(iv) *$|V(P_i)| \geq q$ for $i \in [s]$.*

Proof. Let $\mathfrak{P}$ be the set of all maximal cliques with order at least q in G. If there exists a vertex $x \in V(G)$ such that $|N(x) - \bigcup_{i=1}^{s}\{y \mid x, y \in V(P_i)\}| \geq R(c,\lambda,q)$. By Lemma 4.1, there exists a maximal clique $P_{s+1} \notin \mathfrak{P}$ containing x of order at least $\frac{R(c,\lambda,q)-\binom{\lambda^2}{2}(c-1)}{\lambda^2} + 1$. Since $R(c,\lambda,q) := \lambda^2(q-1)+(c-1)\binom{\lambda^2}{2}$, it follows that the order of P_{s+1} is at least q. This contradicts the definition of $\mathfrak{P}$. Therefore, the properties (i) and (iv) hold.

Let $p_1 := \lambda(\lambda-1)+1$, $p_2 := (\lambda-1)(2\lambda-1)+1$, and $p_3 := (\lambda+1)(\lambda-1)^2+1$. It follows that

$$n_2(\lambda, 2, p_3, \tilde{c}) = \tilde{c} + \tilde{c}((\lambda+1)(\lambda-1)^2+2)(\lambda-1) + (\lambda+1)(\lambda-1)^2 + 2,$$

$$n_2(\lambda+1, 1, p_1, \tilde{c}) = \tilde{c}\lambda(\lambda^2-\lambda+2) + (\lambda^2-\lambda+2),$$

$$n_2(2, \lambda, p_2, \tilde{c}) = 2\tilde{c}(\lambda^2-\lambda+1) + 2\lambda^2 - 3\lambda + 3.$$

For $\lambda \geq 2$, $n_2(\lambda+1, 1, p_1, \tilde{c}) \geq n_2(2, \lambda, p_2, \tilde{c})$ holds as $\tilde{c}$ is at least one. Thus, $q \geq \max\{n_1(\lambda), n_2(\lambda+1, 1, p_1, \tilde{c}), n_2(2, \lambda, p_2, \tilde{c}), n_2(\lambda, 2, p_3, \tilde{c}), (\lambda-1)^2+c+1\}$ holds by the condition.

Let $\mathfrak{h}^{(\lambda+1)}$ be the Hoffman graph with a slim vertex adjacent to $\lambda+1$ fat vertices. Let $\mathfrak{h}_{(\lambda)}$ be the Hoffman graph with two fat vertices adjacent to λ slim vertices such that its slim graph is a complete graph. Let $\mathfrak{h}^{(\lambda)}_{(2)}$ be the Hoffman graph consisting of two adjacent slim vertices, one of which is adjacent to exactly λ fat vertices, while the other has no fat neighbors. By Proposition 2.16, G does not contains $G(\mathfrak{h}^{(\lambda+1)}, p_1)$, $G(\mathfrak{h}_{(\lambda)}, p_2)$, or $G(\mathfrak{h}^{(\lambda)}_{(2)}, p_3)$, as an induced subgraph. It follows that, by Proposition 3.7, $\mathfrak{g}(G,q)$ does not contain $\mathfrak{h}^{(\lambda+1)}$, $\mathfrak{h}^{(\lambda)}_{(2)}$, or $\mathfrak{h}_{(\lambda)}$ as an induced Hoffman subgraph for $q \geq \max\{n_1(\lambda), n_2(\lambda, 2, p_3, \tilde{c}), n_2(\lambda+1, 1, p_1, \tilde{c}), n_2(2, \lambda, p_2, \tilde{c}), (\lambda-1)^2+c+1\}$. Since $\mathfrak{g}(G,q)$ does not contain $\mathfrak{h}^{(\lambda+1)}$ as an induced Hoffman subgraph, each vertex is contained in at most λ maximal cliques in $\mathfrak{B}$. Especially, $N(x) = \bigcup_{i=1}^{s}\{y \mid x, y \in V(P_i)\} - \{x\}$ if x is contained in exactly λ maximal cliques in $\mathfrak{B}$, as $\mathfrak{g}(G,q)$ does not contain $\mathfrak{h}^{(\lambda)}_{(2)}$ as an induced Hoffman subgraph. Additionally, since $\mathfrak{g}(G,q)$ does not contain $\mathfrak{h}_{(\lambda)}$ as an induced Hoffman subgraph, any two maximal cliques in $\mathfrak{B}$ intersect in at most $\lambda-1$ vertices. $\square$

Lemma 4.1 can show that every vertex x is contained in a large clique with order at least $\frac{d(x)}{\lambda^2} - f(\lambda, c)$ if the graph is μ-bounded with parameter c and the smallest eigenvalue at least $-\lambda$. However, as an application of Proposition 3.7 and Lemma 4.1, the following theorem states that every vertex x is contained in a large clique with order at least $\frac{d(x)}{\lambda} - \ell(\lambda, c)$ under the same conditions of Lemma 4.1. This also improves the result in [24].

Theorem 4.6. *Let c be a positive integer and $\lambda \geq 2$ be a real number. There exists a positive integer $\ell := \ell(\lambda, c)$ such that if G is a μ-bounded graph with parameter c and $\lambda_{\min}(G) \geq -\lambda$, then each vertex $x \in V(G)$ lies in a maximal clique $C(x)$ of order at least $\frac{d(x)-\ell}{\lceil\lambda\rceil} + 1$.*

Proof. Let $p = 1 + \lceil\lambda\rceil(\lceil\lambda\rceil - 1)$. Let $\tilde{c} := \min\{c, \lceil\lambda\rceil(\lceil\lambda\rceil-1)\}$ and $q := \max\{n_1(\lceil\lambda\rceil), n_2(\lceil\lambda\rceil + 1, 1, p, \tilde{c}), (\lceil\lambda\rceil-1)^2+c+1\} = \max\{\lceil\lambda\rceil^4 - 2\lceil\lambda\rceil^3 + 3\lceil\lambda\rceil^2 - 3\lceil\lambda\rceil + 3, (\lceil\lambda\rceil^2 - \lceil\lambda\rceil + 2)(\tilde{c}\lceil\lambda\rceil + 1)\}$, where $n_1(\lambda)$ and $n_2(\phi, \sigma, p, \tilde{c})$ are defined in Proposition 3.7. By Proposition 2.16, $\lambda_{\min}(G(\mathfrak{h}^{(\lceil\lambda\rceil+1)}, p)) < -\lceil\lambda\rceil$ holds. This implies that G does not contain $G(\mathfrak{h}^{(\lceil\lambda\rceil+1)}, p)$ as an induced subgraph. Thus, the associated Hoffman graph $\mathfrak{g}(G,q)$ does not contain $\mathfrak{h}^{(\lceil\lambda\rceil+1)}$ as an induced Hoffman subgraph by Proposition 3.7. This means that for each vertex $x \in V(G)$, there are at most $\lceil\lambda\rceil$ distinct maximal cliques with order at least q containing x.

Let $\ell = \ell(\lambda, c) := \lfloor\lambda^2\rfloor(q-1) + \binom{\lfloor\lambda^2\rfloor}{2}(c-1) - 1$. Let $\mathscr{C} = \{C_1(x), \ldots, C_s(x)\}$ be the set of all maximal cliques containing x with order at least q. Let $A(x) := N(x) - \bigcup_{i=1}^{s} V(C_i(x))$. We claim that $|A(x)| \leq \ell$. Otherwise $|A(x)| > \ell$, let H be the subgraph of G induced by $A(x) \cup \{x\}$. Note that in H, x is a vertex with degree at least ℓ. By Lemma 4.1, x lies in a clique C in H with order at least q. It is not hard to see that C is not contained in a clique of $\mathscr{C}$, which contradicts the definition of $\mathscr{C}$. Therefore, $|A(x)| \leq \ell$ and $\max\{|V(C_1(x))|, \ldots, |V(C_s(x))|\} \geq \frac{d(x)-\ell}{\lceil\lambda\rceil} + 1$ holds. $\square$

Remark 4.7. *The proof shows that* $\ell(\lambda, c) = \lfloor\lambda^2\rfloor(q-1) + \binom{\lfloor\lambda^2\rfloor}{2}(c-1) - 1$, *where* $q = \max\{c+1+(\lceil\lambda\rceil-1)^2, \lceil\lambda\rceil^4 - 2\lceil\lambda\rceil^3 + 3\lceil\lambda\rceil^2 - 3\lceil\lambda\rceil + 3, (\lceil\lambda\rceil^2 - \lceil\lambda\rceil + 2)(\tilde{c}\lceil\lambda\rceil + 1)\}$ *and* $\tilde{c} = \min\{c, \lceil\lambda\rceil(\lceil\lambda\rceil - 1)\}$.

5 Applications on DRGs with classical parameters

In this section, we give some applications on distance-regular graphs with classical parameters and prove Corollary 1.7, Theorem 1.2, and Theorem 1.3. We denote by $\mathbb{N}$ the set of all non-negative integers

Proof of Corollary 1.7. Let G be a local graph of Γ. It follows that G is an a_1-regular and μ-bounded graph with parameter $c = c_2 - 1 = 3\alpha + 2$ by Equation (1). By Lemma 2.5, the second largest eigenvalue of Γ is $\lambda_1 = \frac{b_1}{2} - 1$. This implies that the smallest eigenvalue of G is at least -3 by Theorem 2.3.

Furthermore, $\beta \leq \min\{a_1 - 36c - 400\tilde{c} - 83, a_1 - 44c + 5\}$, as $\alpha(2^D - 2) \geq \max\{36c + 400\tilde{c} + 84, 44c - 4\}$ and $a_1 = b_0 - b_1 - c_1 = \beta - 1 + \alpha(2^D - 2)$ by Equations (1) and (2). By Lemma 2.5, the smallest eigenvalue of Γ is $-2^D + 1$. Additionally, by Lemma 2.2, every clique in G has order at most β, as the degree of Γ is $k = b_0 = \beta(2^D - 1)$. Therefore, $\mathfrak{g}(G, q)$ is a 2-fat $\{$, $\}$-line Hoffman graph by Theorem 1.6. $\square$

Proposition 5.1. *Let* Γ *be a distance-regular graph with classical parameters* (D, b, α, β) *such that* $b = 2$ *and* $D \geq 12$. *If* $\alpha \leq 9$ *holds, then* $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, 2\}$.

Proof. By Equation (1), $c_2 = 3 + 3\alpha$ and $c_3 = 7(1 + 3\alpha)$. Since c_2 is an integer, so is 3α. Furthermore, $3\alpha \geq 0$ as c_3 is a positive integer. Thus, $\alpha \in \frac{1}{3}\mathbb{N}$. By Proposition 2.1 with $(i, h) = (6, 6)$ and Equation (1),

$$p_{66}^{12} = \frac{c_7c_8c_9c_{10}c_{11}c_{12}}{c_1c_2c_3c_4c_5c_6} = \frac{230674393235(1+63\alpha)(1+127\alpha)(1+255\alpha)(1+511\alpha)(1+1023\alpha)(1+2047\alpha)}{(1+\alpha)(1+3\alpha)(1+7\alpha)(1+15\alpha)(1+31\alpha)}$$

is a non-negative integer. This proposition can be directly verified by using Wolfram Mathematica 13.3, under the conditions that $\alpha \in \frac{1}{3}\mathbb{N}$, $\alpha \leq 9$, and p_{66}^{12} is a non-negative integer. $\square$

The following theorem gives an upper bound for α in terms of b when $D \geq 9$ and $b \geq 2$.

Theorem 5.2. *Let* D *and* b *be positive integers. If* Γ *is a distance-regular graph with classical parameters* (D, b, α, β) *such that* $D \geq 9$ *and* $b \geq 2$, *then* $(b+1)\alpha$ *is a non-negative integer and one of the following holds:*

(i) $\alpha \leq b(b+1)$,

(ii) $b(b+1) < \alpha < b^2(b+1)+1$ *and* $\beta < (b(b+1)^2-\alpha)\begin{bmatrix} D \\ 1 \end{bmatrix}_b + (b+1)^6 b + (\frac{(b+1)^3((b+1)^2+1)}{2}+1)\alpha - b$.

Moreover, $\alpha \leq b^2(b+1)$ *holds if* $D \geq 10$.

Proof. Since $c_2 = (b+1)(\alpha+1)$ is a positive integer, $(b+1)\alpha$ is an integer. Furthermore, $(b+1)\alpha$ is also a non-negative integer as $c_3 = (1+b+b^2)(1+(b+1)\alpha)$ is a positive integer. Thus, $\alpha \in \frac{1}{b+1}\mathbb{N}$. Assume that $b(b+1) < \alpha$ holds. It follows that $b(b+1) + \frac{1}{b+1} \leq \alpha$.

Let G be a local graph of Γ. It is not hard to see that G has n vertices with $n = b_0 = \beta\frac{b^D-1}{b-1}$, and G is w-regular and μ-bounded with parameter c, where $w = a_1 = \beta - 1 + \alpha\frac{b^D-b}{b-1}$ and $c = c_2 - 1 = (b+1)(\alpha+1) - 1$. Note that $w \geq \alpha\frac{b+1}{b-1}(b^{D-1}-1)$ holds by Inequality (3). By Theorem 2.3, $\lambda_{\min}(G) \geq -b-1$ holds. Let $\ell(\lambda, c)$ be the function defined in Remark 4.7, where $\lambda = b+1$. For $b \geq 2$, $\ell(\lambda, c) < (b+1)^6 b + \frac{(b+1)^3((b+1)^2+1)}{2}\alpha$ holds. By Theorem 4.6, there exists a maximal clique C with order $r \geq \frac{w-\ell(\lambda,c)}{b+1} + 1$. It follows that $r - b^2 \geq g(D, b, \alpha)$ as $w \geq \alpha\frac{b+1}{b-1}(b^{D-1}-1)$, where $g(D, b, \alpha) := \alpha(\frac{(b^{D-1}-1)}{b-1} - \frac{(b+1)^2((b+1)^2+1)}{2}) - (b+1)^5 b - (b-1)(b+1)$.

We claim that each vertex outside C has at most $b(b+1)$ neighbors in C. By Lemma 3.6, each vertex outside C has either at most $b(b+1)$ neighbors in C or at least $r - b^2$ neighbors in C. As $\alpha \geq b(b+1) + \frac{1}{b+1}$, by using Wolfram Mathematica 13.3 under the conditions $b \geq 2$ and $D \geq 9$, it can be obtained that $r - b^2 \geq g(D, b, \alpha) \geq c_2 = (1+\alpha)(1+b)$. Since G is μ-bounded with parameter $c_2 - 1$, each vertex outside C has at most $b(b+1)$ neighbors in C.

Let e be the number of edges between $V(C)$ and $V(G) - V(C)$ in G. It follows that $e = r(w-r+1) \leq b(b+1)(n-r)$. Hence,

$$\beta(r - (b+1)b\frac{b^D-1}{b-1}) \leq r(r - b(b+1) - \alpha\frac{b^D-b}{b-1}).$$

We claim that $r < b(b+1)\frac{b^D-1}{b-1}$. Otherwise $r \geq b(b+1)\frac{b^D-1}{b-1}$, and thus $r \leq \beta$ holds by Lemma 2.2. It follows that $r(r-(b+1)b\frac{b^D-1}{b-1}) \leq r(r - b(b+1) - \alpha\frac{b^D-b}{b-1})$. This implies that $\alpha \leq b(b+1)$, which contradicts the assumption. Hence, $r < b(b+1)\frac{b^D-1}{b-1}$ holds.

Since $b(b+1)\frac{b^D-1}{b-1} \geq r \geq \frac{w-\ell(\lambda,c)}{b+1} + 1$ and $w = \beta - 1 + \alpha\frac{b^D-b}{b-1}$, by Inequality (3), the following inequality holds

$$b(b+1)^2\frac{b^D-1}{b-1} + \ell(\lambda, c) - b - 1 \geq w = \beta - 1 + \alpha\frac{b^D-b}{b-1} \geq \alpha\frac{(b+1)(b^D-b)}{b(b-1)}.$$

It follows that

$$\beta < (b(b+1)^2 - \alpha)\frac{b^D-1}{b-1} + (b+1)^6 b + (\frac{(b+1)^3((b+1)^2+1)}{2} + 1)\alpha - b, \tag{4}$$

and

$$\alpha < b^2(b+1) + f(D, b), \tag{5}$$

where $f(D, b) := \frac{b(b+1)^7}{2b^{D-1} - b(b+1)^4}$. Note that $f(D, b)$ is a monotonically decreasing function with respect to both D and b if $D \geq 9$ and $b \geq 2$.

Now we show that $\alpha \leq b^2(b+1)$ in case of $D \geq 9$ and $2 \leq b \leq 99$. Note that $f(9, b) < 6$ in this case, verified by Wolfram Mathematica 13.3. Thus, $f(D, b) \leq f(9, b) < 6$ and $\alpha < b^2(b+1) + 6$ in this case. By Equation (1) and Proposition 2.1 with $(i, h) = (4, 4)$, p^8_{44} is a non-negative integer depending on α and b. By using Wolfram Mathematica 13.3 under the conditions $\alpha < b^2(b+1)+6$, $2 \leq b \leq 99$, and $D \geq 9$ to verify $p^8_{44} \in \mathbb{N}$, we have $\alpha \leq b^2(b+1)$ in this case.

To prove (ii), it is sufficient to show $\alpha < b^2(b+1) + 1$ in case of $b \geq 100$ and $D \geq 9$. Note that $f(D, b) \leq f(D, 100) < 1$ in this case. It follows that $\alpha < b^2(b+1) + 1$ for $b \geq 100$ and $D \geq 9$, and thus (ii) holds.

To complete the proof, we will show that $\alpha \leq b^2(b+1)$ in case of $b \geq 100$ and $D \geq 10$. Note that $(b+1)f(D, b)$ is a monotonically decreasing function with respect to both D and b if $D \geq 10$ and $b \geq 2$. Thus, $(b+1)f(D, b) \leq 101 f(D, 100) < 1$ in this case. It follows that $f(D, b) < \frac{1}{b+1}$ in this case. This implies that $\alpha \leq b^2(b+1)$ in this case by Inequality (5), as $\alpha \in \frac{1}{b+1}\mathbb{N}$. This completes the proof. $\square$

Proof of Theorem 1.1. It can be directly derived from Theorem 5.2. $\square$

Now, we are in the position to prove Theorem 1.2.

Proof of Theorem 1.2. Let $c = 3\alpha + 2$ and $\tilde{c} := \min\{c, 6\}$. Assume that $\alpha > 2$ holds. By Proposition 5.1, $\alpha \geq 9$ holds. Thus, $\alpha(2^D - 2) \geq \max\{36c + 400\tilde{c} + 84, 44c - 4\}$. Let G be a local graph of Γ. It is not hard to see that G has n vertices with $n = b_0 = \beta(2^D - 1)$, and G is w-regular and μ-bounded with parameter c, where $w = a_1 = \beta - 1 + \alpha(2^D - 2)$. Furthermore, $\mathfrak{g}(G, q)$ is a 2-fat $\{$, $\}$-line Hoffman graph by Corollary 1.7.

By Lemma 3.5, there exists a vertex u in G such that $|N_G(u) \cap N_G(v)| \leq 9$ for each vertex v non-adjacent to u. Let $x \in N_G(u)$ such that $|N_G(x) \cap N_G(u)| = \max\{|N_G(v) \cap N_G(u)| \mid v \in N_G(u)\}$. Let e be the number of edges between $N_G(u) - \{x\}$ and $V(G) - N_G(u) - \{u\}$ in G. It follows that

$$e \leq 9(n - w - 1) = 9(\beta - \alpha)(2^D - 2). \tag{6}$$

By Proposition 3.3 (i), there exists a set of cliques $\mathfrak{B} = \{C_1, \ldots, C_s\}$ in G such that u, v are contained in at least one clique C_i for each vertex $v \in N_G(u) - \{x\}$. By Proposition 3.3 (iii), if there are two cliques $C, C' \in \mathfrak{B}$ such that $\{u, v\} \subset C \cap C'$, then v must be the vertex x. Thus, for each vertex $v \in N_G(u) - \{x\}$, u, v are contained in exactly one clique $C_i \in \mathfrak{B}$. Let $\mathfrak{C}(w) := \{C \in \mathfrak{B} | w \in C\}$ for each vertex w in G. By Proposition 3.3 (ii) and (iii), $|N_G(u) \cap N_G(v)| \leq |C_i| - 2 + \sum_{A \in \mathfrak{C}(u) - \{C_i\}} \sum_{B \in \mathfrak{C}(v) - \{C_i\}} |A \cap B| \leq |C_i| - 2 + 2 \times 2 \times 2$ for each vertex $v \in N_G(u) - \{x\}$. By Lemma 2.5, the smallest eigenvalue of Γ is $-2^D + 1$. As the degree of Γ is $\beta(2^D - 1)$, by Lemma 2.2, every clique in G has order at most β. Thus, $|N_G(u) \cap N_G(v)| \leq \beta + 6$ for each vertex

$v \in N_G(u) - \{x\}$, and

$$e \geq (w-1)(w-\beta-6) = (\beta - 2 + \alpha(2^D-2))(\alpha(2^D-2)-7). \tag{7}$$

Combining Inequalities (6) and (7), $\alpha \leq 9 + \frac{7}{2^D-2} - \frac{9(\alpha(2^D-1)-2)}{\beta-2+\alpha(2^D-2)}$ holds. By Theorem 5.2, $\beta \leq 14(2^D-1)$ holds for $\alpha \geq 9$ and $D > 9$. It follows that $\frac{7}{2^D-2} < \frac{9(\alpha(2^D-1)-2)}{\beta-2+\alpha(2^D-2)}$. This implies $\alpha < 9$, which contradicts the assumption. $\square$

Using a method similar to the proof for Proposition 5.1, we examine the possible values of α for $b \in [100]$ and prove Theorem 1.3.

Proof of Theorem 1.3. Theorem 5.2 shows that $\alpha \in \frac{1}{b+1}\mathbb{N}$ and $\alpha \leq b^2(b+1)$. Furthermore, by Proposition 2.1 and Equation (1), p^{14}_{77}, p^{12}_{66}, p^{10}_{55}, p^{8}_{44}, and p^{6}_{33} all are non-negative integers depending on α and b. This result can be directly verified by using Wolfram Mathematica 13.3, under the conditions that $\alpha \in \frac{1}{b+1}\mathbb{N}$, $\alpha \leq b^2(b+1)$, $b \in [100]$, and p^{14}_{77}, p^{12}_{66}, p^{10}_{55}, p^{8}_{44}, p^{6}_{33} all are non-negative integers only depending on α and b. $\square$

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