A 2-distance set with 277 points in the Euclidean space of dimension 23

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Abstract. We construct a 2-distance set with 277 points in the 23-dimensional Euclidean space $\mathbb{R}^{23}$ having distances 2 and $\sqrt{6}$.

1 Introduction

An s-distance set in a Euclidean space $\mathbb{R}^d$ is a point set such that pairwise distances between these points admit only s different values. Classification and construction of few distance sets in Euclidean space goes back to the pioneering work of Blokhuis [1, 2]. He showed that the size of a 2-distance set in the d-dimensional Euclidean space $\mathbb{R}^d$ is bounded by $\binom{d+2}{2}$. Lisoněk [8] determined the sizes of the largest 2-distance sets in dimensions $d \leq 8$. Since the set of midpoints of edges of a regular d-simplex is a 2-distance set of size $\binom{d+1}{2}$, the main research interest lies in the existence problem of a 2-distance set whose size exceeds $\binom{d+1}{2}$. No construction of such sets has been known for $d > 8$.

In this paper, we construct a 2-distance set in $\mathbb{R}^{23}$ of size 277, one larger than the lower bound $\binom{24}{2}$. Nozaki and Shinohara [9] considered maximal 2-distance sets containing a regular simplex, but the sets they produced have smaller size in dimension 23. Glazyrin and Yu [6] determined that, if a 2-distance set lies in the unit sphere $S^{22} \subseteq \mathbb{R}^{23}$, then its size is bounded by 276. Note that, in addition to the standard example of the set of $\binom{24}{2}$ midpoints of edges of a regular 23-simplex, there are many other 2-distance sets in S^{22} with the same size. The unique regular two-graph on 276 vertices (see [7], and see [4, Section 1.5] for more information on two-graphs) gives a set of equiangular lines in $\mathbb{R}^{23}$, and any set of representing unit vectors forms a 2-distance set of size 276. We show that one of the sets of representing unit vectors can be used to construct a maximal 2-distance set in $\mathbb{R}^{23}$ of size 277, which cannot be extended to a larger 2-distance set in $\mathbb{R}^{23}$.

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2 The Construction

Let $\mathbb{R}^d$ be the d-dimensional linear space over $\mathbb{R}$ equipped with the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^{d} u_i v_i$. As usual, we define the square norm $\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle$.

In [7, Theorem 3.4], Goethals and Seidel provided a graph $\overline{\Gamma}$ in the switching class of the unique regular two-graph on 276 vertices, whose Seidel matrix $S = J - I - 2A(\overline{\Gamma})$ has spectrum

$$\mathrm{Spec}(S) = \{[55]^{23}, [-5]^{253}\},$$

where J is the all-ones matrix and $A(\overline{\Gamma})$ is the adjacency matrix of $\overline{\Gamma}$. The complement Γ of $\overline{\Gamma}$ can be constructed directly as follows. Let

$$X = \{(a, i) \mid a \in \mathbb{F}_3,\ 1 \leq i \leq 11\},$$

and we consider X naturally as the complete multipartite graph with 11 parts, each of which has 3 vertices. Let C be the ternary Golay code, and let $Y = C^{\perp}$, the dual code of C. A description of C and $C^{\perp}$ can be found in [7, Lemma 4.3 and Remark 4.4]. Since $\dim C = 6$, we have $|Y| = 243$. We construct the graph Γ with vertex set $X \cup Y$ by joining $(a, i) \in X$ and $y \in Y$ by an edge if $y_i \neq a$, and also joining $y \in Y$ and $y' \in Y$ by an edge if $y - y'$ has weight 6, that is, $y - y'$ has 6 out of 11 nonzero coordinates. Now $S = 2A(\Gamma) + I - J$ is the desired Seidel matrix.

From the construction, together with the fact that $C^{\perp}$ has 132 vectors of weight 6, we see that the partition $X \cup Y$ of the vertex set of Γ is equitable, with quotient matrix

$$\begin{bmatrix} 30 & 162 \\ 22 & 132 \end{bmatrix}.$$

Thus, $\theta_1 = 81 + \sqrt{6165}$ and $\theta_2 = 81 - \sqrt{6165}$ are two eigenvalues of $A(\Gamma)$.

For each $\alpha \in \{55, -5\}$, let V_α be the eigenspace of S corresponding to the eigenvalue α, and set $W_\alpha := V_\alpha \cap \mathbf{1}^{\perp}$, where $\mathbf{1}$ denotes the all-one vector. Since $\dim W_\alpha \geq \dim V_\alpha - 1$, it follows that Γ has spectrum $\{[27]^{22}, [-3]^{252}, [\theta_1]^1, [\theta_2]^1\}$. Therefore, $A(\Gamma) + 3I$ is positive semi-definite with rank 24, and there exists a set of vectors $\mathbf{V} = \{\mathbf{v}_u \mid u \in X \cup Y\}$ in $\mathbb{R}^{24}$ satisfying $\langle \mathbf{v}_u, \mathbf{v}_v \rangle = (A(\Gamma) + 3I)_{uv}$ for $u, v \in X \cup Y$. It follows that $\mathbf{V}$ forms a 2-distance set in $\mathbb{R}^{24}$ consisting of 276 points

$$\begin{aligned} \|\mathbf{v}\|^2 &= 3 \qquad (\mathbf{v} \in \mathbf{V}), \\ \|\mathbf{v} - \mathbf{v}'\|^2 &\in \{4, 6\} \quad (\mathbf{v}, \mathbf{v}' \in \mathbf{V},\ \mathbf{v} \neq \mathbf{v}'). \end{aligned} \tag{1}$$

In what follows, we identify the vertices of Γ with the corresponding vectors in $\mathbf{V}$. Then for $\mathbf{u}, \mathbf{v} \in \mathbf{V} = X \cup Y$,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \begin{cases} 3 & \text{if } \mathbf{u} = \mathbf{v}, \\ 1 & \text{if } \mathbf{u} \text{ and } \mathbf{v} \text{ are adjacent in } \Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

In [5, Lemma 4.1], Cao *et al.* found a vector $\mathbf{r} \in \mathbb{R}^{24}$ such that $\langle \mathbf{r}, \mathbf{r} \rangle = 2$ and $\langle \mathbf{r}, \mathbf{v} \rangle = 1$ for all $\mathbf{v} \in \mathbf{V}$. The vector $\mathbf{r}$ is called the *switching root*. More explicitly, $\mathbf{r}$ can be defined as follows. Let $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} \subset X$ be the set of vectors corresponding to one of the 11 parts of the complete multipartite graph on X. Then these three vectors are pairwise orthogonal. Define

$$\mathbf{r} = \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 - \frac{4}{33} \sum_{\mathbf{x} \in X} \mathbf{x} + \frac{1}{81} \sum_{\mathbf{y} \in Y} \mathbf{y}.$$

It can be easily verified that $\mathbf{V}$ is contained in the affine hyperplane $\{\mathbf{v} \in \mathbb{R}^{24} \mid \langle \mathbf{v}, \mathbf{r} \rangle = 1\} \cong \mathbb{R}^{23}$ in $\mathbb{R}^{24}$.

Define

$$\mathbf{u} = \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 - \mathbf{r}.$$

It follows that $\langle \mathbf{r}, \mathbf{u} \rangle = 1$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 5$.

Theorem 1. *The set $Z := \{\mathbf{u}\} \cup X \cup Y$ is a 2-distance set of size 277 with squared distances $\{4, 6\}$, contained in the affine hyperplane $\{\mathbf{v} \in \mathbb{R}^{24} \mid \langle \mathbf{v}, \mathbf{r} \rangle = 1\} \cong \mathbb{R}^{23}$.*

Proof. In view of (1), the set $X \cup Y$ is a 2-distance set of size 276 with squared distances $\{4, 6\}$. It suffices to prove

$$\|\mathbf{u} - \mathbf{z}\|^2 \in \{4, 6\} \quad (\mathbf{z} \in X \cup Y).$$

Note that $\langle \mathbf{u}, \mathbf{u} \rangle = 5$ and

$$\begin{aligned} \langle \mathbf{u}, \mathbf{z} \rangle &= \langle \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 - \mathbf{r}, \mathbf{z} \rangle \\ &= \begin{cases} 2 & \text{if } \mathbf{z} \in X, \\ 1 & \text{if } \mathbf{z} \in Y. \end{cases} \end{aligned}$$

It follows that

$$\begin{aligned} \|\mathbf{u} - \mathbf{z}\|^2 &= 5 + 3 - 2\langle \mathbf{u}, \mathbf{z} \rangle \\ &= \begin{cases} 4 & \text{if } \mathbf{z} \in X, \\ 6 & \text{if } \mathbf{z} \in Y. \end{cases} \end{aligned}$$

Therefore, $\{\mathbf{u}\} \cup X \cup Y$ is a 2-distance set having squared distances $\{4, 6\}$. □

We remark that the vector $\mathbf{u}$ is independent of the choice of the part. This can be shown in a more general setting.

Lemma 2. *Let $\mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathbb{R}^d$ be vectors whose Gram matrix is $\begin{bmatrix} nI & J \\ J & nI \end{bmatrix}$, where J denotes the $n \times n$ matrix with all entries equal to 1. Then*

$$\sum_{i=1}^{n} \mathbf{a}_i = \sum_{i=1}^{n} \mathbf{b}_i.$$

Proof. By the Gram matrix, we have

$$\begin{aligned}\left\|\sum_{i=1}^{n}\mathbf{a}_i-\sum_{i=1}^{n}\mathbf{b}_i\right\|^2 &= \left\|\sum_{i=1}^{n}\mathbf{a}_i\right\|^2+\left\|\sum_{i=1}^{n}\mathbf{b}_i\right\|^2-2\sum_{i=1}^{n}\sum_{j=1}^{n}\langle\mathbf{a}_i,\mathbf{b}_j\rangle\\ &= n^2+n^2-2n^2\\ &= 0.\end{aligned}$$

□

Setting $n=3$ in the above lemma, we see that $\mathbf{u}=\mathbf{x}'_1+\mathbf{x}'_2+\mathbf{x}'_3-\mathbf{r}$ holds whenever $\{\mathbf{x}'_1,\mathbf{x}'_2,\mathbf{x}'_3\}\subseteq X$ is a set of one of the 11 parts.

3 Maximality

In this section, we show that our 2-distance set Z of size 277 in the affine hyperplane $H=\{\mathbf{v}\in\mathbb{R}^{24}\mid\langle\mathbf{v},\mathbf{r}\rangle=1\}\cong\mathbb{R}^{23}$ is maximal, in the sense that it cannot be extended to a larger 2-distance set in $\mathbb{R}^{23}$. However, the 2-distance set Z is not maximal in $\mathbb{R}^{24}$, since $Z\cup\{\frac{1+\sqrt{3}}{2}\mathbf{r}\}$ is a larger 2-distance set. Indeed, for $\mathbf{z}\in Z$, we have

$$\begin{aligned}\|\mathbf{z}-\frac{1+\sqrt{3}}{2}\mathbf{r}\|^2 &= \|\mathbf{z}\|^2+2+\sqrt{3}-(1+\sqrt{3})\langle\mathbf{z},\mathbf{r}\rangle\\ &= \|\mathbf{z}\|^2+1\\ &= \begin{cases}6 & \text{if } \mathbf{z}=\mathbf{u},\\ 4 & \text{otherwise.}\end{cases}\end{aligned}$$

It turns out that this is the only way to add a vector in $\mathbb{R}^{24}$ to Z to maintain the 2-distance property, as shown in the following proposition.

Proposition 3. *The 2-distance set Z constructed above is not contained in any larger 2-distance set in H. The only 2-distance set in $\mathbb{R}^{24}$ containing Z properly is $Z\cup\{\frac{1+\sqrt{3}}{2}\mathbf{r}\}$.*

Proof. We only prove the second statement, as the first statement follows from the second. Let $Z'=Z-\mathbf{u}=\{\mathbf{z}-\mathbf{u}\mid\mathbf{z}\in Z\}$. It suffices to show that the only 2-distance set in $\mathbb{R}^{24}$ which properly contains Z' is $Z'\cup\{\frac{1+\sqrt{3}}{2}\mathbf{r}-\mathbf{u}\}$. Since every pair of vectors in Z' has integer inner product, the $\mathbb{Z}$-span M of Z' is an integral lattice in $\mathbf{r}^\perp\cong\mathbb{R}^{23}$.

Suppose that there exists a vector $\mathbf{w}\in\mathbb{R}^{24}$ such that $Z'\cup\{\mathbf{w}\}$ is a 2-distance set properly containing Z'. Since $0\in Z'$, we have $\langle\mathbf{w},\mathbf{w}\rangle=4$ or 6. Let v be the orthogonal projection of $\mathbf{w}$ to $\mathbf{r}^\perp$. Then $\|\mathbf{v}\|^2\le\|\mathbf{w}\|^2\le 6$ and, for $\mathbf{z}'\in Z'$,

$$\langle\mathbf{z}',\mathbf{v}\rangle=\langle\mathbf{z}',\mathbf{w}\rangle$$

$$
\begin{aligned}
&= \frac{1}{2}(\|\mathbf{z}'\|^2 + \|\mathbf{w}\|^2 - \|\mathbf{z}' - \mathbf{w}\|^2) \\
&\in \begin{cases} 2 + \frac{1}{2}(\|\mathbf{z}'\|^2 - \{4,6\}) & \text{if } \|\mathbf{w}\| = 4, \\ 3 + \frac{1}{2}(\|\mathbf{z}'\|^2 - \{4,6\}) & \text{if } \|\mathbf{w}\| = 6 \end{cases} \\
&= \begin{cases} \{\frac{1}{2}\|\mathbf{z}'\|^2, \frac{1}{2}\|\mathbf{z}'\|^2 - 1\} & \text{if } \|\mathbf{w}\| = 4, \\ \{\frac{1}{2}\|\mathbf{z}'\|^2, \frac{1}{2}\|\mathbf{z}'\|^2 + 1\} & \text{if } \|\mathbf{w}\| = 6. \end{cases} \qquad (2)
\end{aligned}
$$

In particular, $\langle \mathbf{z}', \mathbf{v} \rangle \in \mathbb{Z}$ for all $\mathbf{z}' \in Z'$, so $\mathbf{v}$ belongs to the dual lattice

$$M^* = \{\mathbf{x} \in \mathbf{r}^\perp \mid \forall \mathbf{z}' \in M,\ \langle \mathbf{x}, \mathbf{z}' \rangle \in \mathbb{Z}\}$$

of M. By Magma [3], it can be verified that M^* contains 16689170 nonzero vectors $\mathbf{v}$ with $\|\mathbf{v}\|^2 \le 6$. Among them, the only vector $\mathbf{v}$ that satisfies (2) for all $\mathbf{z}' \in Z'$ is $\mathbf{v} = \frac{1}{2}\mathbf{r} - \mathbf{u}$, in which case $\mathbf{w} = \frac{1+\sqrt{3}}{2}\mathbf{r} - \mathbf{u}$. It follows that $Z' \cup \{\mathbf{w}\}$ is the unique 2-distance set in $\mathbb{R}^{24}$ containing Z'. $\square$

Lisoněk [8] showed that there exists a 2-distance set in the Euclidean space $\mathbb{R}^7$ consisting of $29 = \binom{8}{2} + 1$ points. Since $277 = \binom{24}{2} + 1$, our set may be regarded as an analogue of Lisoněk's example in $\mathbb{R}^{23}$. Also, Lisoněk [8, Theorem 4.4] showed that a 2-distance set of size 29 in $\mathbb{R}^7$ is unique up to isometry and scaling. It would be interesting to know if the analogous statement holds for a 2-distance set of size 277 in $\mathbb{R}^{23}$.

Lisoněk [8] also constructed a 2-distance set of size $45 = \binom{10}{2}$ in $\mathbb{R}^8$, which contains the 2-distance set of size 29 in $\mathbb{R}^7$ as a subset. Proposition 3 shows that our 277 points cannot be extended to $325 = \binom{26}{2}$ points in $\mathbb{R}^{24}$ as a 2-distance set.

It remains as a challenging problem to decide whether there exists a 2-distance set in $\mathbb{R}^{24}$ of size $325 = \binom{26}{2}$.

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Data Availability Statement

No data was used for the research described in the article.

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A Magma code

```
Z:=Integers();
F:=GF(3);
C:=GolayCode(F,false);
X:=[ < a,b > : a in F, b in [1..11] ];
#X eq 33;
EX:={ {i,j} : i,j in {1..33}
  | i lt j and X[i][2] ne X[j][2] };
K11x3:=MultipartiteGraph([3:i in [1..11]]);
tf:=IsIsomorphic(Graph< 33 | EX >,K11x3);
tf;

Y:=[ y : y in Dual(C) ];
#Y eq 243;

EXY:={ {i,j+33} : i in {1..33}, j in {1..243}
  | Y[j][X[i][2]] ne X[i][1] };
EY:={ {i+33,j+33} : i,j in {1..243}
  | Weight(Y[i]-Y[j]) eq 6 };
```

```
Gamma:=Graph< 276 | EX join EXY join EY >;
VX:={ Vertices(Gamma) | i : i in {1..33} };
VY:={ Vertices(Gamma) | i : i in {34..276} };
[ [ { #(Neighbours(x) meet VX) : x in VX },
  { #(Neighbours(x) meet VY) : x in VX } ],
  [ { #(Neighbours(y) meet VX) : y in VY },
  { #(Neighbours(y) meet VY) : y in VY } ] ]
  eq [ [ {30},{162} ], [ {22},{132} ] ];

J:=Matrix(Z,276,276,[1:i in [1..276^2]]);
I:=ScalarMatrix(276,1);
A:=AdjacencyMatrix(Gamma);
RX<x>:=PolynomialRing(Z);
RX!CharacteristicPolynomial(A)
  eq (x-27)^22*(x+3)^252*(x^2-162*x+396);
S:=2*A+I-J;
Eigenvalues(S) eq { <55, 23>, <-5, 253> };

gram,XX,r:=LLLGram(A+3*I);
Xi:=XX^(-1);
L:=LatticeWithGram(Submatrix(gram,[1..r],[1..r]));
X276:=[ L![ Xi[i,j] : j in [1..r] ] : i in [1..276] ];
roots:=&join{ {r[1],-r[1]} : r in ShortVectors(L,2,2) };
#roots eq 2;
rt:=[ r : r in roots | (X276[1],r) eq 1 ][1];
&and{ (rt,x) eq 1 : x in X276 };
&and{ (x-y,x-y) in {4,6} : x,y in X276 | x ne y };
u:=&+[ X276[i] : i in [1..3] ]-rt;
(u,u) eq 5 and (rt,u) eq 1;
&and{ (u-x,u-x) in {4,6} : x in X276 };
XL:=[ X276[i] : i in [1..33] ];
YL:=[ X276[i] : i in [34..276] ];
&and{ (XL[i],XL[j]) eq 0 : i,j in [1..3] | i ne j };
rhs:=XL[1]+XL[2]+XL[3]-4/33*&+XL+1/81*&+YL;
Parent(rhs)!rt eq rhs;
Y276:={ x-u : x in X276 };
M1:=sub< L | Y276 >;
Rank(M1) eq 23;
MD:=Dual(M1:Rescale:=false);
Y276D:={ MD | y : y in Y276 };
Minimum(MD) eq 5/2;
Ms:=ShortVectors(MD,5/2,6);
2*#Ms eq 16689170;
```

```
adm4:=func< v | forall(z){ z : z in Y276D
  | (z,v) in { i,i-1 } where i:=(z,z)/2 } >;
not &or{ adm4(w[1]) or adm4(-w[1]) : w in Ms };

adm6:=func< v | forall(z){ z : z in Y276D
  | (z,v) in { i,i+1 } where i:=(z,z)/2 } >;
&join{ { v : v in { w[1],-w[1] } | adm6(v) } : w in Ms }
  eq { 1/2*rt-u };

// Total time: 788.600 seconds, Total memory usage: 4584.38MB
```